



Bers and Hénon, Painlevé and Schroedinger

Serge Cantat

► To cite this version:

Serge Cantat. Bers and Hénon, Painlevé and Schroedinger. Duke Mathematical Journal, 2009, 149 (3), pp.411-460. hal-00186562v2

HAL Id: hal-00186562

<https://hal.science/hal-00186562v2>

Submitted on 5 Dec 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

BERS AND HÉNON, PAINLEVÉ AND SCHRÖDINGER

SERGE CANTAT

ABSTRACT. In this paper, we pursue the study of the holomorphic dynamics of mapping class groups on 2-dimensional character varieties, as initiated in [16, 36]. We shall show that the dynamics of pseudo-Anosov mapping classes resembles in many ways the dynamics of Hénon mappings, and then apply this idea to answer open questions concerning the geometry of discrete and faithful representations, Painlevé sixth equation, and discrete Schrödinger operators.

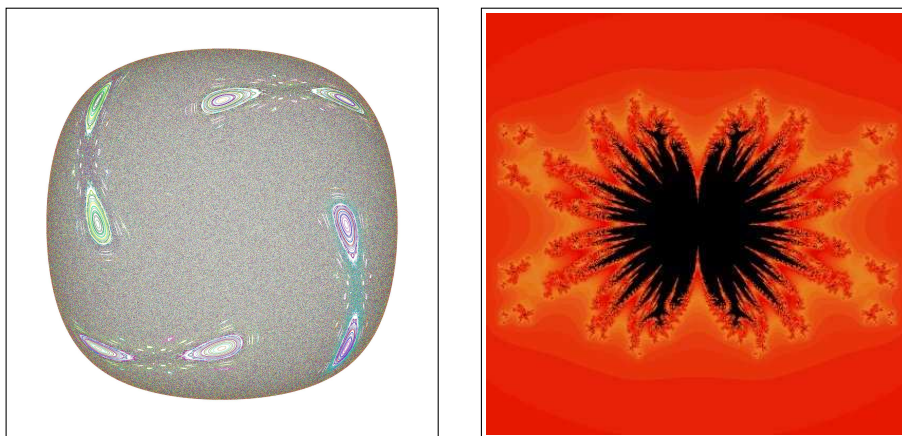


FIGURE 1. Dynamics on character surfaces. Left: Dynamics on the real part of a cubic surface. Right: A slice of the set of complex points with bounded orbit.

CONTENTS

1. Introduction	2
2. The character variety of the four punctured sphere and its automorphisms	6
3. Elements with positive entropy	12
4. The quasi-fuchsian locus and its complement	17

Date: 2007.

5. Real Dynamics of Hyperbolic Elements	21
6. Schrödinger operators and Painlevé equations	35
References	40

1. INTRODUCTION

1.1. Character variety and dynamics. Let \mathbb{T}_1 be the once punctured torus. Its fundamental group is isomorphic to the free group $F_2 = \langle \alpha, \beta \mid \emptyset \rangle$, the commutator of α and β corresponding to a simple loop around the puncture. Since any representation $\rho : F_2 \rightarrow \mathrm{SL}(2, \mathbb{C})$ is uniquely determined by $\rho(\alpha)$ and $\rho(\beta)$, the set $\mathrm{Rep}(\mathbb{T}_1)$ of representations of $\pi_1(\mathbb{T}_1)$ into $\mathrm{SL}(2, \mathbb{C})$ is isomorphic to $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. The group $\mathrm{SL}(2, \mathbb{C})$ acts on this set by conjugation, preserving the three traces

$$x = \mathrm{tr}(\rho(\alpha)), \quad y = \mathrm{tr}(\rho(\beta)), \quad z = \mathrm{tr}(\rho(\alpha\beta)).$$

It turns out that the map $\chi : \mathrm{Rep}(\mathbb{T}_1) \rightarrow \mathbb{C}^3$, defined by $\chi(\rho) = (x, y, z)$, realizes an isomorphism between the algebraic quotient $\mathrm{Rep}(\mathbb{T}_1) // \mathrm{SL}(2, \mathbb{C})$, where $\mathrm{SL}(2, \mathbb{C})$ acts by conjugation, and the complex affine space \mathbb{C}^3 . This quotient will be referred to as the *character variety of the once punctured torus*.

The automorphism group $\mathrm{Aut}(F_2)$ acts by composition on $\mathrm{Rep}(\mathbb{T}_1)$, and induces an action of the mapping class group

$$\mathrm{MCG}^*(\mathbb{T}_1) = \mathrm{Out}(F_2) = \mathrm{GL}(2, \mathbb{Z})$$

on the character variety \mathbb{C}^3 by polynomial diffeomorphisms. Since the conjugacy class of the commutator $[\alpha, \beta]$ is invariant under $\mathrm{Out}(F_2)$, this action preserves the level sets of the polynomial function $\mathrm{tr}(\rho[\alpha, \beta]) = x^2 + y^2 + z^2 - xyz - 2$. As a consequence, for each complex number D , we get a morphism from $\mathrm{Out}(F_2)$ to the group $\mathrm{Aut}(S_D)$ of polynomial diffeomorphisms of the surface S_D , defined by

$$x^2 + y^2 + z^2 = xyz + D.$$

The goal of this paper is to describe the dynamics of mapping classes on these surfaces, both on the complex surface $S_D(\mathbb{C})$ and on the real surface $S_D(\mathbb{R})$ when D is a real number. More generally, we shall study the dynamics of mapping classes on the character variety of the 4-punctured sphere, but we restrict ourselves to the simpler case of the punctured torus in the introduction.

1.2. Hénon type dynamics. Let us fix an element f of the mapping class group $\text{MCG}^*(\mathbb{T}_1)$, that we view simultaneously as a matrix M_f in $\text{GL}(2, \mathbf{Z}) = \text{Out}(F_2)$ or as a polynomial automorphism, still denoted f , of the affine space $\chi(\mathbb{T}_1) = \mathbf{C}^3$ preserving the family of cubic surfaces S_D . Let $\lambda(f)$ be the spectral radius of M_f , so that f is pseudo-Anosov if and only if $\lambda(f) > 1$.

In [16] (see also [36]), it is proved that the topological entropy of $f : S_D(\mathbf{C}) \rightarrow S_D(\mathbf{C})$ is equal to $\log(\lambda(f))$ for all choices of D . The dynamics of mapping classes with zero entropy is described in details in [29, 16]. In section 3, we shall show that the dynamics of pseudo-Anosov classes resembles the dynamics of Hénon automorphisms of the complex plane: All techniques from holomorphic dynamics that have been developed for Hénon automorphisms can be applied to understand the dynamics of mapping classes on character varieties !

As a corollary of this principle, we shall get a positive answer to three different questions. The first one concerns quasi-fuchsian groups and the geometry of the quasi-fuchsian set. The second one concerns the spectrum of certain discrete Schrödinger operators, while the third question is related to Painlevé sixth equation.

1.3. Quasi-Fuchsian spaces and a question of Goldman and Dumas. First, we answer positively a question of Goldman and Dumas (see problem 3.5 in [31]), that we now describe.

When the parameter D is equal to 2, the trace of $\rho[\alpha, \beta]$ vanishes, so that the representations ρ with $\chi(\rho)$ in $S_2(\mathbf{C})$ send the commutator $[\alpha, \beta]$ to an element of order 4 in $\text{SL}(2, \mathbf{C})$. This means that the surface S_2 corresponds in fact to representations of the group $G = \langle \alpha, \beta \mid [\alpha, \beta]^4 \rangle$. Let DF be the subset of $S_2(\mathbf{C})$ corresponding to discrete and faithful representations of G . Some of these representations are fuchsian: These representations $F_2 \rightarrow \text{SL}(2, \mathbf{R})$ come from the existence of hyperbolic metrics on \mathbb{T}_1 with an orbifold point of angle π at the puncture. The interior of DF corresponds to quasi-fuchsian deformations of those fuchsian representations.

Let us now consider the set of conjugacy classes of representations $\rho : G \rightarrow \text{SU}(2)$. This set coincides with the compact connected component of $S_2(\mathbf{R})$. Typical representations into $\text{SU}(2)$ have a dense image and, in this respect, are quite different from discrete faithful representations into $\text{SL}(2, \mathbf{C})$. The following theorem shows that orbits of the mapping class group may contain both types of representations in their closure.

Theorem 1.1. *Let G be the finitely presented group $\langle \alpha, \beta \mid [\alpha, \beta]^4 \rangle$. There exists a representation $\rho : G \rightarrow \mathrm{SL}(2, \mathbb{C})$, such that the closure of the orbit of its conjugacy class $\chi(\rho)$ under the action of $\mathrm{Out}(F_2)$ contains both*

- *the conjugacy class of at least one discrete and faithful representation $\rho' : G \rightarrow \mathrm{SL}(2, \mathbb{C})$, and*
- *the whole set of conjugacy classes of $SU(2)$ -representations of the group G .*

This result answers positively and precisely the question raised by Dumas and Goldman. The strategy of proof is quite general and leads to many other examples; one of them is given in §4.3. The representation ρ' is very special; it corresponds to certain discrete representations provided by Thurston's hyperbolization theorem for mapping tori with pseudo-Anosov monodromy. The same idea may be used to describe DF in dynamical terms (see section 4). To sum up, holomorphic dynamics turns out to be useful to understand the quasi-fuchsian locus and its Bers parametrization.

1.4. Real dynamics, discrete Schrödinger operators, and Painlevé VI equation. The fact that the dynamics of mapping classes is similar to the dynamics of Hénon automorphisms will prove useful to study the real dynamics of mapping classes, i.e. the dynamics of f on the real part $S_D(\mathbb{R})$ when D is a real number. The following theorem, which is the main result of section 5, provides a complete answer to a conjecture popularized by Kadanoff some twenty five years ago (see [38], p. 1872, for a somewhat weaker question). We refer to papers of Casdagli and Roberts (see [17] and [44]), and references therein for a nice mathematical introduction to the subject.

Theorem 1.2. *Let D be a real number. If $f \in \mathrm{MCG}^*(\mathbb{T}_1)$ is a pseudo-Anosov mapping class, the topological entropy of $f : S_D(\mathbb{R}) \rightarrow S_D(\mathbb{R})$ is bounded from above by $\log(\lambda(f))$, and the five following properties are equivalent*

- *the topological entropy of $f : S_D(\mathbb{R}) \rightarrow S_D(\mathbb{R})$ is equal to $\log(\lambda(f))$;*
- *all periodic points of $f : S_D(\mathbb{C}) \rightarrow S_D(\mathbb{C})$ are contained in $S_D(\mathbb{R})$;*
- *the topological entropy of $f : S_D(\mathbb{R}) \rightarrow S_D(\mathbb{R})$ is positive and the dynamics of f on the set $K(f, \mathbb{R}) = \{m \in S_D(\mathbb{R}) \mid (f^n(m))_{n \in \mathbb{Z}} \text{ is bounded} \}$ is uniformly hyperbolic;*
- *the surface $S_D(\mathbb{R})$ is connected*
- *the real parameter D is greater than or equal to 4.*

The main point is the fact that the dynamics is uniformly hyperbolic when $D \geq 4$. As we shall explain in section 6, this may be used to study the spectrum of discrete Schrödinger operators, the potential of which is generated by a primitive substitution: For example, we shall show that the Hausdorff dimension of the spectrum is positive but strictly less than 1 (see section 6 for precise results). This gives also examples of Painlevé VI equations with nice and rich monodromy (see section 6.2), thereby answering a question of Iwasaki and Uehara in [35].

TABLE 1. Dynamics of pseudo-Anosov classes on $S_D(\mathbf{R})$

values of parameter	real part of S_D	dynamics on $K(f, \mathbf{R})$
$D < 0$	four disks	$K(f, \mathbf{R}) = \emptyset$
$D = 0$	four disks and a point	$K(f, \mathbf{R}) = \{(0, 0, 0)\}$
$0 < D < 4$	four disks and a sphere	non uniformly hyperbolic
$D = 4$	the Cayley cubic	uniformly hyperbolic
$D > 4$	a connected surface	uniformly hyperbolic

1.5. Organization of the paper. As mentioned above, we shall study the dynamics of the mapping class group of the four punctured sphere on its character variety; this includes the case of the once punctured torus as a particular case. Section 2 summarizes known useful results, fixes the notations, and describes the dynamics of mapping classes at infinity. Section 3 establishes a dictionary between the Hénon case and the case of character varieties, listing important consequences regarding the dynamics of mapping classes. This is applied in section 4 to study the quasi-fuchsian space. Section 5 describes the dynamics of mapping classes on the real algebraic surfaces $S_D(\mathbf{R})$, for $D \in \mathbf{R}$. This is certainly the most involved part of this paper. It requires a translation of most known facts for Hénon automorphisms to the case of character varieties, and a study of one parameter families of real polynomial automorphisms with maximal entropy. The proof of theorem 1.2, which is given in sections 5.2 and 5.3, could also be used in the study of families of Hénon mappings. We then apply theorem 1.2 to the study of Schrödinger operators and Painlevé VI equations in section 6.

1.6. Acknowledgement. This paper greatly benefited from discussions with Frank Loray, with whom I collaborated on a closely related article (see [16]). I also want to thank Eric Bedford, Cliff Earle, Bill Goldman, Katsunori Iwasaki, Robert MacKay, Yair Minsky, John Smillie, Takato Uehara and Karen Vogtmann for illuminating talks and useful discussions. Most of the content of this paper has been written while I was visiting Cornell University in 2006/2007, and part of it was already described during a conference of the ACI "Systèmes Dynamiques Polynomiaux" in 2004: I thank both institutions for their support.

2. THE CHARACTER VARIETY OF THE FOUR PUNCTURED SPHERE AND ITS AUTOMORPHISMS

This section summarizes known results concerning the character variety of a four punctured sphere and the action of its mapping class group on this algebraic variety. Most of these results can be found in [9], [36], and [16].

2.1. The sphere minus four points. Let \mathbb{S}_4^2 be the four punctured sphere. Its fundamental group is isomorphic to a free group of rank 3,

$$\pi_1(\mathbb{S}_4^2) = \langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma\delta = 1 \rangle,$$

where the four homotopy classes α , β , γ , and δ correspond to loops around the puncture. Let $\text{Rep}(\mathbb{S}_4^2)$ be the set of representations of $\pi_1(\mathbb{S}_4^2)$ into $\text{SL}(2, \mathbb{C})$. Let us associate the 7 following traces to any element ρ of $\text{Rep}(\mathbb{S}_4^2)$,

$$\begin{aligned} a &= \text{tr}(\rho(\alpha)) & b &= \text{tr}(\rho(\beta)) & c &= \text{tr}(\rho(\gamma)) & d &= \text{tr}(\rho(\delta)) \\ x &= \text{tr}(\rho(\alpha\beta)) & y &= \text{tr}(\rho(\beta\gamma)) & z &= \text{tr}(\rho(\gamma\alpha)). \end{aligned}$$

The polynomial map $\chi : \text{Rep}(\mathbb{S}_4^2) \rightarrow \mathbb{C}^7$ defined by $\chi(\rho) = (a, b, c, d, x, y, z)$ is invariant under conjugation, by which we mean that $\chi(\rho') = \chi(\rho)$ if ρ' is conjugate to ρ by an element of $\text{SL}(2, \mathbb{C})$, and it turns out that the algebra of polynomial functions on $\text{Rep}(\mathbb{S}_4^2)$ which are invariant under conjugation is generated by the components of χ . Moreover, the components of χ satisfy the quartic equation

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D, \quad (2.1)$$

in which the variables A , B , C , and D are given by

$$\begin{aligned} A &= ab + cd, & B &= ad + bc, & C &= ac + bd, \\ \text{and } D &= 4 - a^2 - b^2 - c^2 - d^2 - abcd. \end{aligned} \quad (2.2)$$

In other words, the algebraic quotient $\chi(\mathbb{S}_4^2) := \text{Rep}(\mathbb{S}_4^2) // \text{SL}(2, \mathbb{C})$ of $\text{Rep}(\mathbb{S}_4^2)$ by the action of $\text{SL}(2, \mathbb{C})$ by conjugation is isomorphic to the six-dimensional quartic hypersurface of \mathbb{C}^7 defined by equation (2.1).

The affine algebraic variety $\chi(\mathbb{S}_4^2)$ is called the “character variety of \mathbb{S}_4^2 ”. For each choice of four complex parameters A, B, C , and D , $S_{(A,B,C,D)}$ (or S is there is no obvious possible confusion) will denote the cubic surface of \mathbb{C}^3 defined by the equation (2.1). The family of surfaces $S_{(A,B,C,D)}$, with A, B, C , and D describing \mathbb{C} , will be denoted by Fam.

2.2. Automorphisms and the modular group Γ_2^* . The (extended) mapping class group of \mathbb{S}_4^2 acts on $\chi(\mathbb{S}_4^2)$ by polynomial automorphisms: This defines a morphism

$$\begin{cases} \text{Out}(\pi_1(\mathbb{S}_4^2)) & \rightarrow & \text{Aut}(\chi(\mathbb{S}_4^2)) \\ \Phi & \mapsto & f_\Phi \end{cases}$$

such that $f_\Phi(\chi(\rho)) = \chi(\rho \circ \Phi^{-1})$ for any representation ρ .

The group $\text{Out}(\pi_1(\mathbb{S}_4^2))$ contains a copy of $\text{PGL}(2, \mathbb{Z})$ which is obtained as follows. Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ be a torus and σ be the involution of \mathbb{T} defined by $\sigma(x, y) = (-x, -y)$. The fixed point set of σ is the 2-torsion subgroup of \mathbb{T} . The quotient \mathbb{T}/σ is homeomorphic to the sphere, \mathbb{S}^2 , and the quotient map $\pi : \mathbb{T} \rightarrow \mathbb{T}/\sigma = \mathbb{S}^2$ has four ramification points, corresponding to the four fixed points of σ . The group $\text{GL}(2, \mathbb{Z})$ acts linearly on \mathbb{T} and commutes with σ . This yields an action of $\text{PGL}(2, \mathbb{Z})$ on the sphere \mathbb{S}^2 , which permutes the ramification points of π . Taking these four ramification points as the punctures of \mathbb{S}_4^2 , we get a morphism

$$\text{PGL}(2, \mathbb{Z}) \rightarrow \text{MCG}^*(\mathbb{S}_4^2),$$

that turns out to be injective, with finite index image (see [10, 16]). As a consequence, $\text{PGL}(2, \mathbb{Z})$ acts by polynomial transformations on $\chi(\mathbb{S}_4^2)$.

Let Γ_2^* be the subgroup of $\text{PGL}(2, \mathbb{Z})$ whose elements coincide with the identity modulo 2. This group coincides with the stabilizer of the fixed points of σ , so that Γ_2^* acts on \mathbb{S}_4^2 and fixes its four punctures. Consequently, Γ_2^* acts polynomially on $\chi(\mathbb{S}_4^2)$ and preserves the fibers of the projection

$$(a, b, c, d, x, y, z) \mapsto (a, b, c, d).$$

From this we obtain, for any choice of four complex parameters (A, B, C, D) , a morphism from Γ_2^* to the group $\text{Aut}(S_{(A,B,C,D)})$ of polynomial diffeomorphisms of the surface $S_{(A,B,C,D)}$.

Theorem 2.1 ([25, 16]). *For any choice of A, B, C , and D , the morphism*

$$\Gamma_2^* \rightarrow \text{Aut}(S_{(A,B,C,D)})$$

is injective and the index of its image is bounded from above by 24. For a generic choice of the parameters, this morphism is an isomorphism.

The area form Ω , which is globally defined by the formulas

$$\Omega = \frac{dx \wedge dy}{2z + xy - C} = \frac{dy \wedge dz}{2x + yz - A} = \frac{dz \wedge dx}{2y + zx - B}$$

on $S \setminus \text{Sing}(S)$, is almost invariant under the action of Γ_2^* , by which we mean that $f^*\Omega = \pm\Omega$ for any f in Γ_2^* (see [16]). In particular, the dynamics of mapping classes on each surface S is conservative.

2.3. Compactification and automorphisms. Let S be any member of the family Fam. The closure \bar{S} of S in $\mathbb{P}^3(\mathbb{C})$ is given by the cubic homogeneous equation

$$w(x^2 + y^2 + z^2) + xyz = w^2(Ax + By + Cz) + Dw^3.$$

As a consequence, one easily proves that the trace of \bar{S} at infinity does not depend on the parameters and coincides with the triangle Δ given by the equations

$$xyz = 0, \quad w = 0,$$

and, moreover, that the surface \bar{S} is smooth in a neighborhood of Δ (all singularities of \bar{S} are contained in S). The three sides of Δ are the lines $D_x = \{x = 0, w = 0\}$, $D_y = \{y = 0, w = 0\}$ and $D_z = \{z = 0, w = 0\}$; the vertices are $v_x = [1 : 0 : 0 : 0]$, $v_y = [0 : 1 : 0 : 0]$ and $v_z = [0 : 0 : 1 : 0]$. The “middle points” of the sides are respectively $m_x = [0 : 1 : 1 : 0]$, $m_y = [1 : 0 : 1 : 0]$, and $m_z = [1 : 1 : 0 : 0]$.

Since the equation defining S is of degree 2 with respect to the x variable, each point (x, y, z) of S gives rise to a unique second point (x', y, z) . This procedure determines a holomorphic involution of S , namely $s_x(x, y, z) = (A - yz - x, y, z)$. Geometrically, the involution s_x corresponds to the following: If m is a point of \bar{S} , the projective line which joins m and the vertex v_x of the triangle Δ intersects \bar{S} on a third point; this point is $s_x(m)$. The same construction provides two more involutions s_y and s_z , and therefore a subgroup

$$\mathcal{A} = \langle s_x, s_y, s_z \rangle$$

of the group $\text{Aut}(S)$ of polynomial automorphisms of the surface S . It is proved in [16] that the group \mathcal{A} coincides with the image of Γ_2^* into $\text{Aut}(S)$, that is obtained by the action of $\Gamma_2^* \subset \text{MCG}^*(\mathbb{S}_4^2)$ on the character variety $\chi(\mathbb{S}_4^2)$. More precisely, s_x , s_y , and s_z correspond respectively to the automorphisms determined by the following elements of Γ_2^*

$$r_x = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \quad r_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_z = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}.$$

In particular, there is no non trivial relations between the three involutions s_x , s_y and s_z , so that \mathcal{A} is isomorphic to the free product of three copies of $\mathbf{Z}/2\mathbf{Z}$.

Summing up, *the image of the mapping class group in $\text{Aut}(S_{(A,B,C,D)})$, the group Γ_2^* , and \mathcal{A} correspond to finite index subgroups of the full automorphism group of $\text{Aut}(S_{(A,B,C,D)})$.* This is the reason why we shall focus on the dynamics of $\Gamma_2^* = \mathcal{A}$ on the surfaces $S \in \text{Fam}$.

2.4. Notations and remarks. The conjugacy class of a representation $\rho : \pi_1(\mathbb{S}_4^2) \rightarrow \text{SL}(2, \mathbf{C})$ will be denoted $[\rho]$. In general, this conjugacy class is uniquely determined by its image $\chi(\rho)$ in the character variety $\chi(\mathbb{S}_4^2)$, and we shall identify $\chi(\rho)$ to $[\rho]$.

Automorphisms of surfaces $S_{(A,B,C,D)}$ will be denoted by standard letters, like f, g, h, \dots ; the group \mathcal{A} will be identified to its various realizations as subgroups of $\text{Aut}(S_{(A,B,C,D)})$, where (A,B,C,D) describes \mathbf{C}^4 . If M is an element of Γ_2^* , the automorphism associated to M is denoted f_M ; this provides an isomorphism between Γ_2^* and each realization of \mathcal{A} . If f is an automorphism of $S_{(A,B,C,D)}$ which is contained in \mathcal{A} , M_f will denote the unique element of Γ_2^* which corresponds to f .

If $\Phi \in \text{MCG}^*(\mathbb{S}_4^2)$ is a mapping class, the associated automorphism of the character variety will be denoted by f_Φ .

The character surfaces S_D that appeared in the introduction in the case of the once punctured torus are isomorphic to $S_{(0,0,0,D)}$ by a simultaneous multiplication of the variables by -1 . As a consequence, the study of the dynamics on all character surfaces $S \in \text{Fam}$ contains the case of the once punctured torus.

2.5. Dynamics at infinity. The group \mathcal{A} also acts by birational transformations of the compactification \bar{S} of S in $\mathbb{P}^3(\mathbf{C})$. In this section, we describe the dynamics at infinity, i.e. on the triangle Δ .

If f is an element of \mathcal{A} , the birational transformation of \bar{S} defined by f is not everywhere defined. The set of its indeterminacy points is denoted by $\text{Ind}(f)$; f is said to be *algebraically stable* if f^n does not contract any curve onto $\text{Ind}(f)$ for $n \geq 0$ (see [46, 21] for this notion).

The group Γ_2^* acts by isometries on the Poincaré half plane \mathbb{H} . Let j_x, j_y and j_z be the three points on the boundary of \mathbb{H} with coordinates $-1, 0$, and ∞ respectively. The three generators r_x, r_y , and r_z of Γ_2^* (see 2.3) are the reflections of \mathbb{H} around the three geodesics which join respectively j_y to j_z , j_z to j_x , and j_x to j_y . As a consequence, Γ_2^* coincides with the group

of symmetries of the tessellation of \mathbb{H} by ideal triangles, one of which has vertices j_x , j_y and j_z . This picture will be useful to describe the action of \mathcal{A} on Δ .

First, one shows easily that the involution s_x acts on the triangle Δ in the following way: The image of the side D_x is the vertex v_x and the vertex v_x is blown up onto the side D_x ; the sides D_y and D_z are invariant and s_x permutes the vertices and fixes the middle points m_y and m_z of each of these sides. An analogous statement holds of course for s_y and s_z . In particular, the action of \mathcal{A} at infinity does not depend on the set of parameters (A, B, C, D) .

Beside the three involutions s_x , s_y and s_z , three new elements of \mathcal{A} play a particular role in the study of the dynamics of $\text{MCG}^*(\mathbb{S}_4^2)$. These elements are

$$g_x = s_z \circ s_y, \quad g_y = s_x \circ s_z, \quad \text{and} \quad g_z = s_y \circ s_x.$$

They correspond to Dehn twists in the mapping class group. Each of them preserves one of the coordinate variables x , y or z respectively. The action of g_x (resp. g_y , resp. g_z) on Δ is the following: g_x contracts both D_y and $D_z \setminus \{v_y\}$ on v_z , and preserves D_x ; its inverse contracts D_y and $D_z \setminus \{v_z\}$ on v_y . In particular $\text{Ind}(g_x) = v_y$ and $\text{Ind}(g_x^{-1}) = v_z$.

Let f be any element of $\mathcal{A} \setminus \{\text{Id}\}$ and M_f be the corresponding element of Γ_2^* . If M_f is elliptic, f is conjugate to s_x , s_y or s_z . If M_f is parabolic, f is conjugate to an iterate of g_x , g_y or g_z (see [16]). In both cases, the action of f on Δ has just been described.

If M_f is hyperbolic, the isometry M_f of \mathbb{H} has two fixed points at infinity, an attracting fixed point $\omega(f)$ and a repulsive fixed point $\alpha(f)$, and the action of f on Δ can be described as follows: The three sides of Δ are blown down on the vertex v_x (resp. v_y resp. v_z) if $\omega(f)$ is contained in the interval $[j_y, j_z]$ (resp. $[j_z, j_x]$, resp. $[j_x, j_y]$); the unique indeterminacy point of f is v_x (resp. v_y resp. v_z) if $\alpha(f)$ is contained in $[j_y, j_z]$ (resp. $[j_z, j_x]$, resp. $[j_x, j_y]$). In particular $\text{Ind}(f)$ coincides with $\text{Ind}(f^{-1})$ if and only if $\alpha(f)$ and $\omega(f)$ are in the same connected component of $\partial\mathbb{H} \setminus \{j_x, j_y, j_z\}$. As a consequence, we get the following result (see [16] for details).

Proposition 2.2. *Let S be any member of the family Fam. Let f be an element of \mathcal{A} . Assume that the element M_f of Γ_2^* that corresponds to f is hyperbolic.*

- *The birational transformation $f : \bar{S} \rightarrow \bar{S}$ is algebraically stable if, and only if f is a cyclically reduced composition of the three involutions s_x , s_y and s_z (in which each involution appears at least once).*

In particular, any hyperbolic element f of \mathcal{A} is conjugate to an algebraically stable element of \mathcal{A} .

- *If f is algebraically stable, f^n contracts the whole triangle $\Delta \setminus \text{Ind}(f)$ onto $\text{Ind}(f^{-1})$ as soon as n is a positive integer.*

2.6. Topological entropy and types of automorphisms. An element f of \mathcal{A} will be termed elliptic, parabolic or hyperbolic, according to the type of the isometry $M_f \in \Gamma_2^*$. By theorem B of [16], the *topological entropy* $h_{\text{top}}(f)$ of $f : S_{(A,B,C,D)}(\mathbf{C}) \rightarrow S_{(A,B,C,D)}(\mathbf{C})$ does not depend on the parameters (A, B, C, D) and is equal to the logarithm of the spectral radius $\lambda(f)$ of M_f :

$$h_{\text{top}}(f) = \log(\lambda(f)). \quad (2.3)$$

In particular, pseudo-Anosov mapping classes are exactly those with positive entropy on the character surfaces $S_{(A,B,C,D)}$. As explained in the previous section, Dehn twists correspond to parabolic elements and are conjugate to a power of g_x , g_y or g_z , while elliptic automorphisms are conjugate to s_x , s_y or s_z .

Remark 2.3. This should be compared to the description of the group of polynomial automorphisms of the affine plane \mathbf{C}^2 . If h is an element $\text{Aut}(\mathbf{C}^2)$, either h is conjugate to an elementary automorphism, which means that h preserves the pencil of lines $y = c^{ste}$, or the topological entropy is equal to $\log(d(h))$, where $d(h)$ is an integer (see §3.2 for references).

2.7. The Cayley cubic. The surface S_4 will play a central role in this paper. This surface is the unique element of Fam with four singularities, and is therefore the unique element of Fam that is isomorphic to the Cayley cubic (see [16]). We shall call it "the Cayley cubic" and denote it by S_C . This surface is isomorphic to the quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the involution $\eta(x, y) = (x^{-1}, y^{-1})$. The map

$$\pi_C(u, v) = \left(u + \frac{1}{u}, v + \frac{1}{v}, uv + \frac{1}{uv} \right)$$

gives an explicit isomorphism between $(\mathbf{C}^* \times \mathbf{C}^*)/\eta$ and S_C : Fixed points of η , as $(-1, 1)$, correspond to singular points of S_C .

The group $\text{GL}(2, \mathbf{Z})$ acts on $\mathbf{C}^* \times \mathbf{C}^*$ by *monomial transformations*: If $M = (m_{ij})$ is an element of $\text{GL}(2, \mathbf{Z})$, and if (u, v) is a point of $\mathbf{C}^* \times \mathbf{C}^*$, then

$$(u, v)^M = (u^{m_{11}} v^{m_{12}}, u^{m_{21}} v^{m_{22}}).$$

This action commutes with η , so that $\text{PGL}(2, \mathbf{Z})$ acts on the quotient S_C .

The surface S_C is one of the character surfaces for the once punctured torus: It corresponds to reducible representations of $\pi_1(\mathbb{T}_1)$ (with $\text{tr}(\rho[\alpha, \beta]) = 2$). Of course, the monomial action of $\text{PGL}(2, \mathbf{Z})$ on S_C coincides with the action of the mapping class group of \mathbb{T}_1 on the character surface S_C . Changing signs of coordinates, it also coincides with the action of $\Gamma_2^* \subset \text{MCG}(\mathbb{S}_4^2)$ on the character surface corresponding to parameters $(a, b, c, d) = (0, 0, 0, 0)$ or $(2, 2, 2, -2)$, up to permutation of a, b, c , and d and multiplication by -1 (see [16]).

The product $\mathbf{C}^* \times \mathbf{C}^*$ retracts by deformation onto the 2-dimensional real torus $\mathbb{S}^1 \times \mathbb{S}^1$. The monomial action of $\text{GL}(2, \mathbf{Z})$ preserves this torus: It acts "linearly" on this torus if we use the parametrization $u = e^{2i\pi s}$, $v = e^{2i\pi t}$. After deleting the four singularities of S_C , the real part $S_C(\mathbf{R})$ has five components, and the closure of the unique bounded component is the image of $\mathbb{S}^1 \times \mathbb{S}^1$ by π_C . The closure of the four unbounded components are images of $\mathbf{R}^+ \times \mathbf{R}^+$, $\mathbf{R}^+ \times \mathbf{R}^-$, $\mathbf{R}^- \times \mathbf{R}^+$, and $\mathbf{R}^- \times \mathbf{R}^-$, by π_C .

3. ELEMENTS WITH POSITIVE ENTROPY

In this section, we describe the dynamics of hyperbolic elements in the group \mathcal{A} on any surface $S_{(A,B,C,D)}(\mathbf{C})$ of our family Fam.

Let f be a hyperbolic element of \mathcal{A} . After conjugation by an element h of \mathcal{A} , we can assume that f is algebraically stable; in our context, this property means that, for any element S of Fam, the indeterminacy set of the birational transformation $\bar{f} : \bar{S} \dashrightarrow \bar{S}$ and the indeterminacy set of \bar{f}^{-1} are two distinct vertices of the triangle at infinity Δ (see §2.5). In what follows, we shall assume that f is algebraically stable and denote $\text{Ind}(f^{-1})$ by v_+ and $\text{Ind}(f)$ by v_- .

3.1. Attracting basin of $\text{Ind}(f^{-1})$. The birational transformation \bar{f} is holomorphic in a neighborhood of v_+ and contracts $\Delta \setminus \{v_-\}$ on v_+ . In particular, \bar{f} contracts the two sides of Δ that contain v_+ on the vertex v_+ . Using the terminology of [26], \bar{f} determines a rigid and irreducible contracting germ near v_+ .

Theorem 3.1. *If f is an algebraically stable hyperbolic element of \mathcal{A} , then there exist an element N_f of $\text{GL}(2, \mathbf{Z})$ with positive entries which is conjugate to M_f in $\text{PGL}(2, \mathbf{Z})$, a neighborhood \mathcal{U} of v_+ in \bar{S} , and a holomorphic diffeomorphism $\Psi_f^+ : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{U}$ such that $\Psi_f^+(0, 0) = v_+$ and*

$$\Psi_f^+((u, v)^{N_f}) = f(\Psi_f^+(u, v))$$

for all (u, v) in the bidisk $\mathbb{D} \times \mathbb{D}$.

Proof. Let \mathcal{U} be a small bidisk around v_+ , in which the two sides of Δ correspond to the two coordinate axis. The fundamental group of $\mathcal{U} \setminus \Delta$ is isomorphic to $(\mathbb{Z}^2, +)$ and \bar{f} induces an automorphism N_f of this group. Since f is a rigid and irreducible contracting germ near v_+ , a theorem of Dloussky and Favre asserts that f is locally conjugate to the monomial transformation that N_f determines ; in particular, f being a local contraction, N_f has positive entries (see class 6 of the classification, Table II, and page 483 in [26]). The fact that the conjugacy Ψ_f is defined on the whole bidisk will be part of the next proposition.

To prove that N_f is conjugate to $\pm M_f$ in $\text{GL}(2, \mathbb{Z})$, one argues as follows. The matrix N_f is obtained from the action of f on the fundamental group of $\mathcal{U} \setminus \Delta$. In the case of the Cayley cubic,

$$\pi_C : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \overline{S_C} \setminus \Delta$$

is a 2 to 1 covering, $\mathbb{C}^* \times \mathbb{C}^*$ retracts by deformation on the torus $\mathbb{S}^1 \times \mathbb{S}^1$, and the action of f on the fundamental group of $\mathcal{U} \setminus \Delta$ is therefore covered by the action of M_f on $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$. This implies that N_f is conjugate to M_f in $\text{PGL}(2, \mathbb{Z})$. Since the general case is obtained from the Cayley case by a smooth deformation, this is true for any set of parameters (A, B, C, D) . \square

Let $s(f)$ be the slope of the eigenline of the linear planar transformation N_f , which corresponds to the eigenvalue $1/\lambda(f)$; $s(f)$ is a negative real number. The basin of attraction of the origin for the monomial transformation N_f is

$$\overline{\Omega(N_f)} = \{(u, v) \in \mathbb{C}^2 \mid |v| < |u|^{s(f)}\}.$$

In particular, this basin contains the full bidisk. We shall denote by $\Omega(N_f)$ the intersection of $\overline{\Omega(N_f)}$ with $\mathbb{C}^* \times \mathbb{C}^*$. Similar notations will be used for the basin of attraction $\overline{\Omega(v_+)}$ of the point v_+ for \bar{f} in \overline{S} , and for its intersection $\Omega(v_+)$ with S .

Proposition 3.2. *The conjugacy Ψ_f^+ extends to a biholomorphism between $\Omega(N_f)$ and $\Omega(v_+)$.*

Proof. Since the monomial transformation N_f is contracting and $f : S \rightarrow S$ is invertible, we can extend Ψ_f^+ to $\Omega(N_f) \cap (\mathbb{C}^* \times \mathbb{C}^*)$ by the functional equation

$$\Psi_f^+(u, v) = f^{-n}(\Psi_f^+((u, v)^{N_f^n}),$$

where n is large enough for $(u, v)^{N_f^n}$ to be in the initial domain of definition of Ψ_f^+ . The map $\Psi_f^+ : \Omega(N_f) \cap (\mathbb{C}^* \times \mathbb{C}^*) \rightarrow S$ is a local diffeomorphism, the

image of which coincides with the basin of attraction of v_+ in S . It remains to prove that the map Ψ_f^+ is injective. Assume that $\Psi_f^+(u_1, v_1) = \Psi_f^+(u_2, v_2)$. Then $f^n(\Psi_f^+(u_1, v_1)) = f^n(\Psi_f^+(u_2, v_2))$, and therefore

$$\Psi_f^+((u_1, v_1)^{N_f^n}) = \Psi_f^+((u_2, v_2)^{N_f^n}),$$

for any n . Since Ψ_f^+ is injective in a neighborhood of the origin, and since the monomial transformation N_f is also injective, this implies $(u_1, v_1) = (u_2, v_2)$. \square

In what follows, $\|\cdot\|$ will denote the usual euclidean norm in \mathbf{C}^3 .

Corollary 3.3. *Let f be an algebraically stable hyperbolic element of \mathcal{A} . If m is a point of S with an unbounded positive orbit, then $f^n(m)$ goes to $\text{Ind}(f^{-1})$ when n goes to $+\infty$ and*

$$\log \|f^n(m)\| \sim \lambda(f)^n.$$

Proof. First we apply the previous results to the study of f^{-1} and its basin of attraction near v_- . Let us fix a small ball B around v_- in the surface \bar{S} . If B is small enough, then B is contained in the basin of attraction of f^{-1} : The orbit of a point $m_0 \in B$ by f^{-1} stays in B and converges towards v_- . Since f contracts $\Delta \setminus \{v_-\}$ on v_+ , there is a neighborhood $\mathcal{V} \subset \bar{S}$ of $\Delta \setminus B$ which is contained in the basin of attraction of v_+ . Let m be a point with unbounded orbit. Since $\mathcal{V} \cup B$ is a neighborhood of Δ , the sequence $(f^n(m))$ will visit $\mathcal{V} \cup B$ infinitely many times. Let n_1 be the first positive time for which $f^{n_1}(m)$ is contained in $\mathcal{V} \cup B$. Let n_2 be the first time after n_1 such that $f^{n_2}(m)$ escapes B . Then $f^n(m)$ never comes back in B for $n > n_2$. Pick a $n > n_2$ such that $f^n(m)$ is contained in $\mathcal{V} \cup B$. Then $f^n(m)$ is in \mathcal{V} and therefore in the basin of v_+ . This implies that the sequence $f^n(m)$ converges towards v_+ . In order to study the growth of $\|f^n(m)\|$ in a neighborhood of v_+ , we apply the conjugacy Ψ_f^+ : What we now need to control is the growth of $\|(u, v)^{N_f^n}\|^{-1}$, and the result is an easy exercise using exponential coordinates $(u, v) = (e^s, e^t)$, in $\mathbb{D}^* \times \mathbb{D}^*$. \square

Corollary 3.4. *If f is a hyperbolic element of \mathcal{A} and A, B, C , and D are four complex numbers, f does not preserve any algebraic curve in $S_{(A,B,C,D)}$.*

Proof. Let us assume the existence of a set of parameters (A, B, C, D) and of an f -invariant algebraic curve $E \subset S_{(A,B,C,D)}$. Let \bar{E} be the Zariski-closure of E in $\bar{S}_{(A,B,C,D)}(\mathbf{C})$; f induces an automorphism \bar{f} of the compact Riemann surface \bar{E} . Since \mathbf{C}^3 does not contain any 1-dimensional compact complex subvariety, \bar{E} contains points at infinity. These points must coincide with v_+

and/or v_- . In particular, the restriction of f to \overline{E} has at least one superattracting (or superrepulsive) fixed point. This is a contradiction with the fact that $\overline{f} : \overline{E} \rightarrow \overline{E}$ is an automorphism. \square

3.2. Bounded orbits and Julia sets. Let us consider the case of a polynomial diffeomorphism h of the affine plane \mathbf{C}^2 with positive topological entropy (an automorphism of Hénon type). After conjugation by an element of $\text{Aut}[\mathbf{C}^2]$, we may assume that h is algebraically stable in $\mathbb{P}^2(\mathbf{C})$. In that case, the dynamics of h at infinity also exhibits two attracting fixed points, one for h , w_+ , and one for h^{-1} , w_- , but there are three differences with the dynamics of hyperbolic elements of \mathcal{A} : The exponential escape growth rate is an integer $d(h)$ (while $\lambda(f)$ is an irrational quadratic integer), the model to which h is conjugate near w_+ is not invertible, and the conjugacy Ψ_h is a covering map of infinite degree between the basins of attraction. We refer the reader to [34], [26] and [33] for an extensive study of this situation.

Beside these differences, we shall see that the dynamics of hyperbolic elements of \mathcal{A} is similar to the dynamics of Hénon automorphisms. In analogy with the Hénon case, let us introduce the following definitions:

- $K^+(f)$ is the set of bounded forward orbits. This is also the set of points m in the surface S , for which $(f^n(m))$ does not converge to v_+ when n goes to $+\infty$.

$K^-(f)$ is the set of bounded backward orbits, and $K(f) = K^+(f) \cap K^-(f)$.

- $J^+(f)$ is the boundary of $K^+(f)$, $J^-(f)$ is the boundary of $K^-(f)$, and $J(f)$ is the subset of $\partial K(f)$ defined by $J(f) = J^-(f) \cap J^+(f)$. The set $J(f)$ will be called *the Julia set of f* .

- $J^*(f)$ is the closure of the set of saddle periodic points of f (see below).

Figure 1, right, is a one dimensional (complex) slice of $K^+(f)$ when $f = s_x \circ s_y \circ s_z$ and $(A, B, C, D) = (0, 0, 0, 0)$. The left part of this figure represents a few hundred orbits of this automorphism on the unique compact connected component of $S_{(0,0,0,2)}(\mathbf{R})$.

3.3. Green functions and dynamics. We define the Green functions of f by

$$G_f^+(m) = \lim_{n \rightarrow +\infty} \frac{1}{\lambda(f)^n} \log^+ \|f^n(m)\|, \quad (3.1)$$

$$G_f^-(m) = \lim_{n \rightarrow +\infty} \frac{1}{\lambda(f)^n} \log^+ \|f^{-n}(m)\|. \quad (3.2)$$

By proposition 3.2 and its corollary, both functions are well defined and the zero set of G_f^\pm coincides with $K^\pm(f)$. Moreover, the convergence is uniform

on compact subsets of S . Since $\log^+ \|\cdot\|$ is a pluri-subharmonic function, G_f^+ (resp. G_f^-) is pluri-subharmonic and is pluri-harmonic on the complement of $K^+(f)$ (resp. $K^-(f)$) (see [5, 27, 46] for the details of the proof). These functions satisfy the invariance properties

$$G_f^+ \circ f = \lambda(f) G_f^+ \quad \text{and} \quad G_f^- \circ f = \lambda(f)^{-1} G_f^- \quad (3.3)$$

The following results have been proved for Hénon mappings; we list them with appropriate references, in which the reader can find a proof which applies to our context (see also [15], [2], [24], [46] for similar contexts).

- G_f^+ and G_f^- are Hölder continuous (see [22], sections 2.2, 2.3). The currents

$$T_f^+ = dd^c G_f^+ \quad \text{and} \quad T_f^- = dd^c G_f^- \quad (3.4)$$

are closed and positive, and $f^* T_f^\pm = \lambda(f)^\pm T_f^\pm$. By [5], section 3, the support of T_f^+ is $J^+(f)$, the support of T_f^- is $J^-(f)$ (see also [46]).

- Since the potentials G_f^+ and G_f^- are continuous, the product

$$\mu_f = T^+ \wedge T^- \quad (3.5)$$

is a well defined positive measure, and is f -invariant. Multiplying G_f^+ and G_f^- by positive constants, we can, and we shall assume that μ_f is a probability measure. (see [5], section 3)

- The topological entropy of f is $\log(\lambda(f))$ and the measure μ_f is the unique f -invariant probability measure with maximal entropy. (see [4], section 3)

- If m is a saddle periodic point of f , its unstable (resp. stable) manifold $W^u(m)$ (resp. $W^s(m)$) is parametrized by \mathbf{C} . Let $\xi : \mathbf{C} \rightarrow S$ be such a parametrization of $W^u(m)$ with $\xi(0) = m$. Let $\mathbb{D} \subset \mathbf{C}$ be the unit disk, and let χ be a smooth non negative function on $\xi(\mathbb{D})$, with $\chi(m) > 0$ and $\chi = 0$ in a neighborhood of $\xi(\partial\mathbb{D})$. Let $[\xi(\mathbb{D})]$ be the current of integration on $\xi(\mathbb{D})$. The sequence of currents

$$\frac{1}{\lambda(f)^n} f_*^{-n} (\chi \cdot [\xi(\mathbb{D})])$$

weakly converges toward a positive multiple of T_f^- . (see [6], sections 2 and 3, [27])

- By corollary 3.4, periodic points of f are isolated. The number of periodic points of period N grows like $\lambda(f)^N$. Most of them are hyperbolic

saddle points: If $\mathcal{P}(f, N)$ denotes either the set of periodic points with period N or the set of periodic saddle points of period N , then

$$\sum_{m \in \mathcal{P}(f, N)} \delta_m \rightarrow \mu_f$$

where the convergence is a weak convergence in the space of probability measures on compact subsets of S . (see [4], [3], and [24])

- The support $J^*(f)$ of μ_f simultaneously coincides with the Shilov boundary of $K(f)$ and with the closure of periodic saddle points of f . In particular, any periodic saddle point of f is in the support of μ_f . If p and q are periodic saddle points, then $J^*(f)$ coincides with the closure of $W^u(p) \cap W^s(q)$. (see [4] and [3])
- Since f is area preserving (see §2.2), the interior of $K(f)$, $K^+(f)$ and $K^-(f)$ coincide. In particular, the interior of $K^+(f)$ is a bounded open subset of $S(\mathbf{C})$. (see lemma 5.5 of [5])

4. THE QUASI-FUCHSIAN LOCUS AND ITS COMPLEMENT

In this section, we shall mostly restrict the study to the case of the once punctured torus with a cusp, and provide hints for more general statements.

4.1. Quasi-fuchsian space and Bers' parametrization. Let \mathbb{T}_1 be a once punctured torus. Let $\text{Teich}(\mathbb{T}_1)$ be the Teichmüller space of complete hyperbolic metrics on \mathbb{T}_1 with finite area 2π , or equivalently with a cusp at the puncture: $\text{Teich}(\mathbb{T}_1)$ is isomorphic, and will be identified, to the upper half plane \mathbb{H}^+ . The dynamics of $\text{MCG}(\mathbb{T}_1)$ on $\text{Teich}(\mathbb{T}_1)$ is conjugate to the usual action of $\text{PSL}(2, \mathbf{Z})$ on \mathbb{H}^+ .

Any point in the Teichmüller space gives rise to a representation $\bar{\rho} : F_2 \rightarrow \text{PSL}(2, \mathbf{R})$ that can be lifted to four distinct representations into $\text{SL}(2, \mathbf{R})$. The cusp condition gives rise to the same equation $\text{tr}(\rho[\alpha, \beta]) = -2$ for any of these four representations. This provides four embeddings of the Teichmüller space into the surface $S_0(\mathbf{R})$: The four images are the four unbounded components of $S_0(\mathbf{R})$, each of which is diffeomorphic to \mathbb{H}^+ ; apart from these four components, $S_0(\mathbf{R})$ contains an isolated singularity at the origin. This point corresponds to the conjugacy class of the representation ρ_{quat} , defined by

$$\rho_{quat}(\alpha) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho_{quat}(\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.1)$$

Its image coincides with the quaternionic group of order eight. The mapping class group of the torus acts on $S_0(\mathbf{R})$, preserves the origin and the connected

component

$$S_0^+(\mathbf{R}) = S_0(\mathbf{R}) \cap (\mathbf{R}^+)^3,$$

and permutes the remaining three components.

Let $DF \subset S_0(\mathbf{C})$ be the set of conjugacy classes of discrete and faithful representations $\rho : F_2 \rightarrow \mathrm{SL}(2, \mathbf{C})$ with $\mathrm{tr}(\rho[\alpha, \beta]) = -2$. This set is composed of four distinct connected components, one of them, DF^+ , containing $S_0^+(\mathbf{R})$. The component $S_0^+(\mathbf{R})$ is made of conjugacy classes of fuchsian representations, and the set QF of their quasi-fuchsian deformations coincides with the interior of DF^+ (see [41], and the references therein).

Let \mathbb{T}'_1 be the once punctured torus with the opposite orientation. Bers' parametrization of the space of quasi-fuchsian representations provides a holomorphic bijection

$$\mathrm{Bers} : \mathrm{Teich}(\mathbb{T}_1) \times \mathrm{Teich}(\mathbb{T}'_1) \rightarrow \mathrm{Int}(DF^+).$$

We may identify $\mathrm{Teich}(\mathbb{T}_1)$ with the upper half plane \mathbb{H}^+ and $\mathrm{Teich}(\mathbb{T}'_1)$ with the lower half plane \mathbb{H}^- . The group $\mathrm{PSL}(2, \mathbf{Z})$ acts on $\mathbb{P}^1(\mathbf{C})$, preserving $\mathbb{P}^1(\mathbf{R})$, \mathbb{H}^+ , and \mathbb{H}^- . In particular, $\mathrm{MCG}(\mathbb{T}_1) = \mathrm{SL}(2, \mathbf{Z})$ acts diagonally on

$$\mathrm{Teich}(\mathbb{T}_1) \times \mathrm{Teich}(\mathbb{T}'_1) = \mathbb{H}^+ \times \mathbb{H}^-.$$

With these identifications, the map Bers conjugates the diagonal action of $\mathrm{MCG}(\mathbb{T}_1)$ on $\mathbb{H}^+ \times \mathbb{H}^-$ with its action on the character variety: If Φ is a mapping class and f_Φ is the automorphism of S_0 which is determined by Φ , then

$$\mathrm{Bers}(\Phi(X), \Phi(Y)) = f_\Phi(\mathrm{Bers}(X, Y))$$

for any (X, Y) in $\mathbb{H}^+ \times \mathbb{H}^-$. It conjugates the action of $\mathrm{MCG}(\mathbb{T}_1)$ on the set

$$\{(z_1, z_2) \in \mathbb{H}^+ \times \mathbb{H}^- \mid z_1 = \overline{z_2}\}$$

with the corresponding action on $S_0^+(\mathbf{R})$. The Bers map extends up to the boundary of $\mathbb{H}^+ \times \mathbb{H}^-$ minus its diagonal (we shall call it the restricted boundary, and denote it by $\partial^*(\mathbb{H}^+ \times \mathbb{H}^-)$). Minsky proved in [40] that Bers induces a continuous bijection from $\partial^*(\mathbb{H}^+ \times \mathbb{H}^-)$ to the boundary of DF^+ .

4.2. Mapping torus and fixed points. Let $\Phi \in \mathrm{MCG}(\mathbb{T}_1)$ be a pseudo-Anosov mapping class. Let X_Φ be the mapping torus determined by Φ : The threefold X_Φ is obtained by suspension of \mathbb{T}_1 over the circle, with monodromy Φ . Thurston's hyperbolization theorem tells us that X_Φ can be endowed with a complete hyperbolic metric of finite volume. This provides a discrete and faithful representation

$$\rho_\Phi : \pi_1(X_f) \rightarrow \mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbf{C})$$

If we restrict ρ_Φ to the fundamental group of the torus fiber of X_Φ , and if we choose the appropriate lift to $\mathrm{SL}(2, \mathbb{C})$, we get a point $[\rho_\Phi]$ in $\mathrm{DF}^+ \subset S_0(\mathbb{C})$ which is fixed by the automorphism f_Φ . Let $\alpha(\Phi)$ (resp. $\omega(\Phi)$) be the repulsive (resp. attracting) fixed point of Φ on the boundary of $\mathrm{Teich}(\mathbb{T}_1)$. Since Bers is a continuous conjugacy, we have

$$\mathrm{Bers}(\alpha(\Phi), \omega(\Phi)) = [\rho_\Phi].$$

The fixed point $(\omega(\Phi), \alpha(\Phi))$ provides a second fixed point on the boundary of DF^+ : This point may be obtained by the same construction with Φ^{-1} in place of Φ . In [39], McMullen proved that $[\rho_\Phi]$ is a hyperbolic fixed point of f_Φ . The stable and unstable manifolds of f_Φ at $[\rho_\Phi]$ intersect DF^+ along its boundary,

$$W^u([\rho_\Phi]) \cap \mathrm{DF}^+ = \mathrm{Bers}(\{\alpha(\Phi)\} \times \overline{\mathbb{H}^-} \setminus \{(\alpha(\Phi), \alpha(\Phi))\}), \quad (4.2)$$

$$W^s([\rho_\Phi]) \cap \mathrm{DF}^+ = \mathrm{Bers}(\overline{\mathbb{H}^+} \times \{\omega(\Phi)\} \setminus \{(\omega(\Phi), \omega(\Phi))\}). \quad (4.3)$$

In particular, the union of all stable manifolds $W^s([\rho_\Phi]) \cap \mathrm{DF}^+$ of f_Φ , where Φ describes the set of pseudo-Anosov mapping classes, form a dense subset of $\partial \mathrm{DF}^+$.

Remark 4.1. Each pseudo-Anosov class Φ determines an automorphism f_Φ , and therefore a subset $K^+(f_\Phi)$ of $S_0(\mathbb{C})$. The complement $\Omega^+(f_\Phi)$ of $K^+(f_\Phi)$ is open: It coincides with the basin of attraction of f_Φ at infinity. The interior of DF^+ is contained in the intersection

$$\Omega(\mathrm{MCG}(\mathbb{T}_1)) := \bigcap_{\Phi} \Omega^+(f_\Phi)$$

where Φ describes the set of pseudo-Anosov classes in the mapping class group $\mathrm{MCG}(\mathbb{T}_1) = \mathrm{SL}(2, \mathbb{Z})$. Since stable manifolds are dense in the boundary of DF , the quasi fuchsian locus $\mathrm{Int}(\mathrm{DF}^+)$ is a connected component of the interior of $\Omega(\mathrm{MCG}(\mathbb{T}_1))$.

4.3. Two examples. We now present two orbits in the complement of DF .

Theorem 4.2. *Let Φ be any pseudo-Anosov mapping class and $[\rho_\Phi]$ be one of the two fixed points of f_Φ on the boundary of the subset DF^+ of $S_0(\mathbb{C})$. There exists a representation $\rho_0 : \pi_1(\mathbb{T}_1) \rightarrow \mathrm{SL}(2, \mathbb{C})$ such that $[\rho_0]$ is an element of $S_0(\mathbb{C})$ and*

- *the sequence $(f_\Phi)^n[\rho_0]$ converges toward the discrete and faithful representation $[\rho_\Phi]$ when n goes to $+\infty$;*
- *the closure of the mapping-class group orbit of $[\rho_0]$ contains the origin $(0,0,0)$ of $S_0(\mathbb{C})$, i.e. the conjugacy class of the finite representation ρ_{quat} .*

Proof. The fixed point $[\rho_\Phi]$ is hyperbolic, with a stable manifold $W^s([\rho_\Phi])$. The origin $(0,0,0)$ is the unique singular point of $S_0(\mathbf{C})$. It corresponds to the representation $[\rho_{quat}]$ which is defined by equation (4.1). This point is fixed by f , and a direct computation shows that the differential of f at the origin has finite order (order 1 or 2).

From section 3.3, the interior of $K^+(f)$ coincides with the interior of $K^-(f)$ and is therefore an f -invariant bounded open subset of $S_0(\mathbf{C})$. In particular, $\text{Int}(K^+(f))$ is Kobayashi hyperbolic, and if $[\rho_q]$ is in $\text{Int}(K^+(f))$, then f is locally linearizable around the origin $[\rho_q]$. Since $Df_{[\rho_q]}$ has finite order, f would have finite order too. This contradiction shows that $[\rho_q]$ is not in the interior of $K^+(f)$.

According to a theorem of Bowditch (see theorem 5.5 of [12]), there exists a neighborhood U of the origin in $S_0(\mathbf{C})$ with the property that any mapping class group orbit starting in U contains the origin in its closure.

We know that $W^s([\rho_f])$ is dense in the boundary of $K^+(f)$ (see §3.3). Since $[\rho_q]$ is in $\partial K^+(f)$, $W^s([\rho_f])$ intersects the Bowditch's neighborhood U . Any point $[\rho_0]$ in $W^s([\rho_f]) \cap U$ satisfies the properties of the theorem. \square

Proof of theorem 1.1. Let us now consider the dynamics of the mapping class

$$\Psi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

on the surface $S_2(\mathbf{C})$. The surface $S_2(\mathbf{C})$ corresponds to representations $\rho : G \rightarrow \text{SL}(2, \mathbf{C})$, where $G = \langle \alpha, \beta | [\alpha, \beta]^4 \rangle$ (see §1.3).

As explained, for example, in [39], section 3.7, the surface $S_2(\mathbf{C})$ contains an f_Ψ -invariant open subset corresponding to quasi-fuchsian deformations of the fuchsian groups obtained by endowing a hyperbolic metric on \mathbb{T}_1 with an orbifold point of angle π at the puncture. Thurston's hyperbolization theorem provides a hyperbolic fixed point $[\rho_\Psi]$ of f_Ψ on the boundary of this set: The representation $\rho_\Psi : G \rightarrow \text{SL}(2, \mathbf{C})$ is discrete and faithful and comes from the existence of a hyperbolic structure on the complement of the figure eight knot, with an orbifold structure along the knot.

The subset of $S_2(\mathbf{C})$ corresponding to conjugacy classes of $\text{SU}(2)$ representations coincides with the unique bounded connected component of $S_2(\mathbf{R})$, and is homeomorphic to a sphere (see [28], figure 4). This component is f_Ψ -invariant, and f_Ψ has exactly two fixed points on it, namely $(x, x/(x-1), x)$, with

$$x = \frac{\sqrt{17} + \sqrt{1 + \sqrt{17}/2}}{2} \quad \text{or} \quad \frac{\sqrt{17} - \sqrt{1 + \sqrt{17}/2}}{2}.$$

Both of them are saddle points. Let $[\rho_{\text{SU}}]$ be one of these fixed points, and let $W^s([\rho_{\text{SU}}])$ and $W^u([\rho_{\Psi}])$ be the stable and unstable manifolds of f_{Ψ} through $[\rho_{\text{SU}}]$.

From section 3.3, we know that $W^s([\rho_{\text{SU}}])$ intersects $W^u([\rho_{\Psi}])$. Let $[\rho_0]$ be one of these intersection points. The f_{Ψ} -orbit of $[\rho_0]$ contains both $[\rho_{\Psi}]$ and $[\rho_{\text{SU}}]$.

Finite orbits of $\text{MCG}(\mathbb{T}_1)$ are listed in [23] and $[\rho_{\text{SU}}]$ does not appear in the list. As a consequence, the mapping class group orbit of $[\rho_{\text{SU}}]$ is infinite and dense in the component of $\text{SU}(2)$ -representations (see [29], [30], or [16]). This implies that the closure of the orbit of $[\rho_{\text{SU}}]$ contains both $[\rho_{\Psi}]$ and the $\text{SU}(2)$ -component of $S_2(\mathbf{R})$. \square

5. REAL DYNAMICS OF HYPERBOLIC ELEMENTS

In this section, we study the dynamics of hyperbolic elements on the real surfaces $S_{(A,B,C,D)}(\mathbf{R})$ when the parameters are real numbers. The main goal of this section is to prove theorem 5.10 below, which extends, and precises, theorem 1.2.

5.1. Maximal entropy. Let us fix a hyperbolic element $f \in \mathcal{A}$. If the parameters (A, B, C, D) are real, we get two dynamical systems: The first one takes place on the complex surface $S(\mathbf{C})$ and its main stochastic properties have been listed in section 3.3; the second one is induced by the restriction of f to the real part $S(\mathbf{R})$. From time to time, we shall use the notation $f_{\mathbf{R}}$ for the restriction of f to $S(\mathbf{R})$. For example, we shall say that $f_{\mathbf{R}}$ has *maximal entropy* if the entropy of $f : S(\mathbf{R}) \rightarrow S(\mathbf{R})$ is equal to the topological entropy of $f : S(\mathbf{C}) \rightarrow S(\mathbf{C})$, i.e. to $\log(\lambda(f))$.

Theorem 5.1. *Let f be a hyperbolic element of \mathcal{A} . If (A, B, C, D) are real parameters, the following conditions are equivalent:*

- (1) $f_{\mathbf{R}}$ has maximal entropy;
- (2) $J^*(f)$ is contained in $S(\mathbf{R})$;
- (3) $K(f)$ is contained in $S(\mathbf{R})$.

In that case, $J^(f) = J(f) = K(f)$.*

This theorem is an easy consequence of the results of section 3.3 (see [4], section 10 for a proof). Our first goal is to prove the following result.

Theorem 5.2. *Let f be a hyperbolic element of \mathcal{A} . If (A, B, C, D) are real parameters such that $S_{(A,B,C,D)}(\mathbf{R})$ is connected, then $f_{\mathbf{R}}$ has maximal entropy.*

Remark 5.3. Benedetto and Goldman studied the various topologies that can occur for $S(\mathbf{R})$. Using (a, b, c, d) parameters (see section 2.1), $S(\mathbf{R})$ is connected if and only (i) none of the parameters a, b, c , and d is contained in the interval $(-2, 2)$ and (ii) the product $abcd$ is negative. In that case, the surface $S(\mathbf{R})$ is homeomorphic to a sphere minus four punctures (see [9]). These conditions on (a, b, c, d) define four arcwise connected subsets of \mathbf{R}^4 , that contain respectively the 8 points $(2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, 2\varepsilon_4)$, with $\varepsilon_i = \pm 1$ and $\prod \varepsilon_i = -1$. All these points correspond to the same surface $S_{(0,0,0,4)}$, i.e. to the Cayley cubic S_C . As a consequence, any connected surface $S(\mathbf{R})$ can be smoothly deformed to the Cayley cubic S_C inside Fam.

Before giving a proof of theorem 5.2, let us review a result of Bowen concerning topological lower bounds for the entropy (see [13]). Let f be a homeomorphism of a marked topological space (X, m) , by which we mean that m is a fixed point of f . Then, f determines an automorphism $f_*: \pi_1(X, m) \rightarrow \pi_1(X, m)$. Let us assume that $\pi_1(X)$ is finitely generated, and fix a finite set $\{\alpha_1, \dots, \alpha_k\}$ of generators for $\pi_1(X)$. The growth rate of f_* is defined to be

$$\lambda(f_*) = \limsup_{n \rightarrow +\infty} \left(\frac{1}{n} \text{diam}(f^n(B)) \right)$$

where diam is the diameter with respect to the word metric (using the generators α_i) and B is the ball of radius 1 with respect to this metric. Bowen's theorem shows that

$$h_{\text{top}}(f) \geq \log(\lambda(f_*))$$

as soon as f is a continuous transformation of a compact manifold. Even though $S(\mathbf{R})$ is not compact, we can apply this theorem because unbounded orbits are contained in the basins of attraction of $\text{Ind}(f^{-1})$ and $\text{Ind}(f)$.

Proof of theorem 5.2. Let us first study the case of the Cayley cubic S_C . This surface is singular, and $S_C(\mathbf{R}) \setminus \text{Sing}(S_C)$ contains a unique bounded component. This component $S_C(\mathbf{R})^0$ is a sphere with four punctures and the dynamics of \mathcal{A} (i.e. Γ_2^*) is covered by the monomial action of Γ_2^* on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbf{C}^* \times \mathbf{C}^*$. As a consequence, for any hyperbolic element f in Γ_2^* , the entropy of f on $S_C(\mathbf{R})^0$ is maximal. Moreover, in that case, the expanding factor $\lambda(f_*)$ coincides with the dynamical degree $\lambda(f)$, and Bowen's inequality is an equality.

If we deform the Cayley cubic in such a way that the surface $S(\mathbf{R})$ is smooth and connected, then $S(\mathbf{R})$ is homeomorphic to a four punctured sphere (the punctures are now at infinity), and the action of f on the fundamental group of $S(\mathbf{R})$ has not been changed along the deformation. As a

consequence, Bowen's inequality gives

$$h_{top}(f_{\mathbf{R}}) \geq \log(\lambda(f))$$

and the conclusion follows from

$$h_{top}(f_{\mathbf{R}}) \leq h_{top}(f_{\mathbf{C}}) = \log(\lambda(f)).$$

This concludes the proof for smooth and connected surfaces $S(\mathbf{R})$ (see remark 5.3). If $S(\mathbf{R})$ is not smooth but is connected, then $S(\mathbf{R})$ is a limit of smooth connected members of the family Fam. By semicontinuity of topological entropy, $f_{\mathbf{R}}$ has maximal entropy (see [42]). \square

Corollary 5.4. *Let a, b, c , and d be four real parameters in $\mathbf{R} \setminus [-2, 2]$, the product of which is negative. Let $\rho : \pi_1(\mathbb{S}_4) \rightarrow \mathrm{SL}(2, \mathbf{C})$ be a representation with boundary traces a, b, c , and d . Let $\Phi \in \mathrm{Aut}(\pi_1(\mathbb{S}_4^2))$ be a pseudo-Anosov automorphism. If $\rho \circ \Phi$ is conjugate to ρ , then ρ is conjugate to a representation into $\mathrm{SL}(2, \mathbf{R})$.*

Proof. Let S be the element of the family Fam that corresponds to the parameters (a, b, c, d) . The assumption on the parameters a, b, c , and d implies that $S(\mathbf{R})$ is connected (see remark 5.3), and that there is no $\mathrm{SU}(2)$ -component (this is obvious if $S(\mathbf{R})$ is smooth, since $\mathrm{SU}(2)$ representations would form a compact component, and this follows from [9] in the singular case).

If $\rho \circ \Phi^{-1}$ is conjugate to ρ , then $\chi(\rho)$ is a fixed point of the automorphism f_{Φ} induced by Φ on the surface S . Since $S(\mathbf{R})$ is connected, $f_{\mathbf{R}}$ has maximal entropy. By theorem 5.1, all periodic points of f are contained in $S(\mathbf{R})$. This implies that ρ is conjugate to an $\mathrm{SL}(2, \mathbf{R})$ -valued representation. \square

5.2. Maximal entropy and quasi-hyperbolicity. Bedford and Smillie recently developed a nice theory for Hénon transformations which extends the notion of quasi-hyperbolicity, a notion that had been previously introduced for the dynamics of rational maps of one complex variable. This theory can be applied to our context in order to study hyperbolic automorphisms with maximal entropy.

5.2.1. Quasi-hyperbolicity. Let $Sadd(f)$ be either the set of periodic saddle points of f or the set $W^u(p) \cap W^s(q)$ where p and q are two periodic fixed points of f (see [7] for more examples). With such a choice, $Sadd(f)$ is f -invariant and its closure coincides with $J^*(f)$ (see §3.3). Each point m of $Sadd(f)$ has a stable manifold $W^s(p)$ and an unstable manifold $W^u(p)$, and we can find two injective immersions $\xi_m^u, \xi_m^s : \mathbf{C} \rightarrow S$ such that $\xi_m^{u/s}(0) = m$,

$\xi_m^{u/s}(\mathbf{C}) = W^{u/s}(m)$, and

$$\max\{G^{+/-}(\xi_m^{u/s}(t)) \mid t \in \mathbb{D}\} = 1,$$

where \mathbb{D} is the unit disk. The parametrization ξ_m^u and ξ_m^s are uniquely determined by this normalization up to a rotation of t . Since $Sadd(f)$ is f -invariant and f sends the unstable manifold at m on the unstable manifold at $f(m)$, there is a non zero complex number $\lambda(m)$ such that

$$f(\xi_m^u(t)) = \xi_{f(m)}^u(\lambda(m)t).$$

The number $\lambda(m)$ depends on the choices made for ξ_m^u and $\xi_{f(m)}^u$ but its modulus $|\lambda(m)|$ only depends on m . Since $G^+ \circ f = \lambda(f)G^+$, we obtain easily the inequality $|\lambda(m)| > 1$ for all $m \in Sadd(f)$.

We shall also need the growth function $\text{gro}_m(r)$ of G^+ along the unstable manifold $W^u(m)$, which is defined by $\text{gro}_m(r) = \max_{|t| \leq r} \{G^+(\xi_m^u(t))\}$, and the uniform growth function

$$\text{Gro}(r) = \sup_{m \in Sadd(f)} \{\text{gro}_m(r)\}.$$

Bedford and Smillie proved in [7], section 1, that the following properties are equivalent:

- (1) the family $\{\xi_m^u \mid m \in Sadd(f)\}$ is a normal family;
- (2) $\text{Gro}(r_0) < \infty$ for some $1 < r_0 < \infty$;
- (3) there exists $\kappa > 1$ such that $|\lambda(m)| \geq \kappa$ for all m in $Sadd(f)$;
- (4) $\exists C, \beta < \infty$ such that $\text{gro}_m(r) \leq Cr^\beta$ for all m in S and $r \geq 1$.

If one, and then all, of these properties is satisfied, f is said to be *quasi-expanding*. If f and f^{-1} are quasi-expanding, then f is said to be *quasi-hyperbolic*.

5.2.2. Maximal entropy. It turns out that real Hénon mappings with maximal entropy are necessarily quasi-hyperbolic (see [7], theorem 4.8 and proposition 4.9). The proof of this result can be applied word by word to our context, and gives rise to the following theorem.

Theorem 5.5 (Bedford Smillie, [7] and [8]). *Let f be a hyperbolic element of \mathcal{A} and S be an element of Fam defined by real parameters (A, B, C, D) . If $f_{\mathbf{R}}$ has maximal entropy, then f is quasi-hyperbolic, and any periodic point m of f is a saddle point, with $|\lambda(m)| \geq \lambda(f)$.*

Corollary 5.6. *Let f be a hyperbolic element of \mathcal{A} and S be an element of Fam defined by real parameters (A, B, C, D) . If $S(\mathbf{R})$ is connected, then $f_{\mathbf{R}}$ has maximal entropy and is quasi-hyperbolic.*

5.2.3. Uniform hyperbolicity and consequences. In a subsequent paper, Bedford and Smillie also obtain a precise obstruction to uniform hyperbolicity. Let $p \in S(\mathbf{R})$ be a saddle periodic point of f . The unstable manifold of p in $S(\mathbf{R})$ is the intersection of $S(\mathbf{R})$ with the complex unstable manifold $W^u(p)$. This real unstable manifold is diffeomorphic to the real line \mathbf{R} , and p disconnects it into two half lines. If one of these half unstable manifolds is contained in the complement of $K^+(f)$, one says that p is *u-one-sided*; *s-one-sided* points are defined in a similar way.

Theorem 5.7 (Bedford Smillie, [8]). *Let f be a hyperbolic element of \mathcal{A} and S be an element of Fam defined by real parameters (A, B, C, D) . If $f_{\mathbf{R}}$ has maximal entropy but $K(f)$ is not a hyperbolic set for f , then*

- *there are periodic saddle points p and q (not necessarily distinct) so that $W^u(p)$ intersects $W^s(q)$ tangentially with order 2 contact ;*
- *p is s-one-sided and q is u-one-sided ;*
- *the restriction of f to $K(f)$ is not expansive.*

Theorem 5.8. *Let f be a hyperbolic element of \mathcal{A} . Let S be a smooth surface in the family Fam which is defined by real parameters (A, B, C, D) . If one of the connected components of $S(\mathbf{R})$ is bounded, then the entropy of $f_{\mathbf{R}}$ is not maximal and f has an infinite number of saddle periodic points in $S(\mathbf{C}) \setminus S(\mathbf{R})$.*

Proof. Let us assume that f has maximal entropy and that $S(\mathbf{R})$ has at least one bounded connected component $S(\mathbf{R})^0$. The existence of a bounded component implies that $S(\mathbf{R})$ has five connected components, four of which are unbounded and homeomorphic to disks, and one, $S(\mathbf{R})^0$, is bounded and homeomorphic to a sphere (see [9]). Being f -invariant and compact, $S(\mathbf{R})^0$ is contained in $K(f)$. Since $f_{\mathbf{R}}$ has maximal entropy, $K(f)$ is contained in $S(\mathbf{R})$. Since $K(f)$ is the support of μ_f (see §3.3), $\mu_f(S(\mathbf{R})^0)$ is a positive number. The ergodicity of μ_f and the f -invariance of $S(\mathbf{R})^0$ now imply that $S(\mathbf{R})^0$ has full μ_f measure. As a consequence, $K(f)$ coincides with $S(\mathbf{R})^0$. Since $S(\mathbf{R})^0$ is compact, there is no one-sided periodic point, and theorem 5.7 implies that $K(f)$ is a hyperbolic set. This means that the dynamics of f on $S(\mathbf{R})^0$ is uniformly hyperbolic. In particular, the unstable directions of f determine a continuous line field on $S(\mathbf{R})^0$, and we get a contradiction because $S(\mathbf{R})^0$ is a sphere. \square

Corollary 5.9. *Let D be a real number and S_D be the element of Fam defined by the real parameters $(0, 0, 0, D)$. The following properties are equivalent:*

(i) *there exists a hyperbolic element f in \mathcal{A} such that $f : S_D(\mathbf{R}) \rightarrow S_D(\mathbf{R})$ has*

maximal entropy, (ii) any hyperbolic element f in \mathcal{A} has maximal entropy on $S_D(\mathbf{R})$, and (iii) $D \geq 4$.

Proof. If $D > 4$, then $S(\mathbf{R})$ is connected and smooth and the result follows from theorem 5.2. If $D \leq 0$, the result follows from the fact that the action of the mapping class group on $S(\mathbf{R})$ is totally discontinuous (see [30]). If $0 < D < 4$, then $S(\mathbf{R})$ has a compact connected component $S(\mathbf{R})^0$ and the conclusion follows from the previous theorem. \square

5.3. Uniform hyperbolicity. We now prove theorem 1.2 in the following more general form.

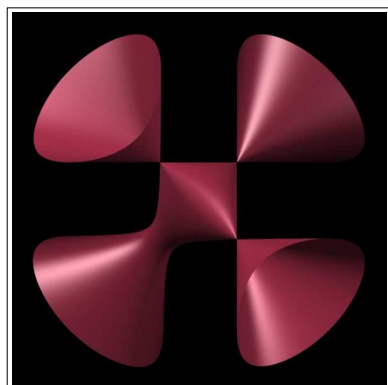
Theorem 5.10. *Let f be a hyperbolic element of \mathcal{A} . Let S be an element of Fam defined by real parameters. If $S(\mathbf{R})$ is connected, then*

- *the entropy of $f_{\mathbf{R}}$ is maximal; its value is $\log(\lambda(f))$;*
- *the set of bounded orbits of $f : S(\mathbf{C}) \rightarrow S(\mathbf{C})$ is a compact subset $K(f)$ of $S(\mathbf{R})$;*
- *the automorphism f admits a unique invariant probability measure μ_f of maximal entropy, and the support of μ_f coincides with $K(f)$; periodic saddle points equidistribute toward μ_f ;*
- *the dynamics of f on $K(f)$ is uniformly hyperbolic.*

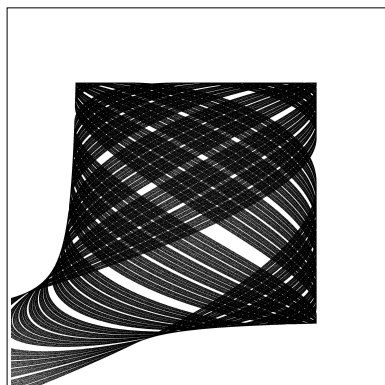
The only property that has not been proven yet is the last one. In fact, we shall prove more than uniform hyperbolicity: Our objective includes a complete description of the complement of $K^+(f)$, in order to explain pictures like the one provided in figure 2. This will be achieved in section 5.4.

5.3.1. Notations and preliminaries. In what follows, we fix a hyperbolic element f of \mathcal{A} , and assume that f preserves orientation (replace f by f^2 if f reverses orientation). We denote by \mathcal{H} the space of real parameters (A, B, C, D) such that $S(\mathbf{R})$ is connected. In order to prove theorem 5.10, and theorem 5.15, we shall study the dynamics of f on all surfaces $S = S_{(A, B, C, D)}$ with (A, B, C, D) in \mathcal{H} . For such surfaces, maximal entropy implies the following properties:

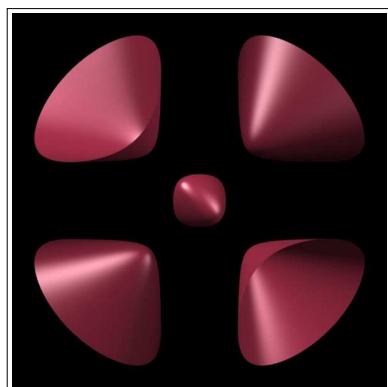
- (1) $K(f)$ coincides with $J(f)$ and is a subset of $S(\mathbf{R})$; moreover, periodic points are hyperbolic, all of them are contained in $K(f)$, and intersections between stable and unstable manifolds are also contained in $K(f)$ (see theorem 5.5) ;
- (2) the set of one-sided points is a finite subset $OS(f)$ of $J(f)$ (see [8], sections 3 and 4) ;



A-1



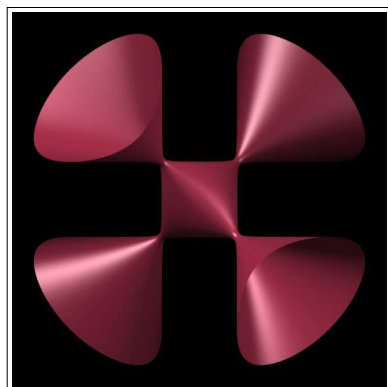
A-2



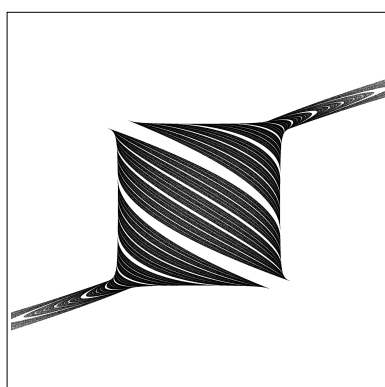
B-1



B-2



C-1



C-2

FIGURE 2. Examples of stable manifolds.

- (3) if m is a point of tangency between stable and unstable manifolds of f , the α and ω -limit sets of m are contained in $OS(f)$ (see theorem 2.7 of [8]) ;
- (4) in the complement of $OS(f)$, stable and unstable manifolds of f form two laminations of $J(f)$ (see proposition 5.3 of [7]) ;
- (5) a tangency between a stable and an unstable manifold is always quadratic (see section 2 in [8], section 5 of [7]).

Once again, as in the proof of theorem 5.2, the main argument is to understand perturbations of the Cayley cubic, *i.e.* perturbations of $f_{\mathbf{R}} : S_C(\mathbf{R}) \rightarrow S_C(\mathbf{R})$.

5.3.2. Conical singularities of the Cayley cubic. The surface S_C has four conical singularities. If we blow up \mathbf{C}^3 at the four singular points of S_C , the strict transform \widehat{S}_C of S_C is smooth, and the singular points are replaced by four projective lines $\mathbb{P}^1(\mathbf{C})$. These projective lines are called exceptional divisors and will be denoted by E_i , $i = 1, 2, 3, 4$. Another way to get the same surface \widehat{S}_C is to blow up $\mathbf{C}^* \times \mathbf{C}^*$ at the four fixed points of the involution $\eta(u, v) = (1/u, 1/v)$. The involution η and the action of the group Γ_2^* can be lifted to $\widehat{\mathbf{C}^* \times \mathbf{C}^*}$, and the quotient $\widehat{\mathbf{C}^* \times \mathbf{C}^*} / \eta$, is isomorphic to \widehat{S}_C .

On the exceptional divisors, each hyperbolic element f of \mathcal{A} now has two hyperbolic fixed points, instead of one singular fixed point (this is the reason why we assume that f preserves orientation; if f reverses orientation, then f has a pair of periodic points of period 2 on each exceptional divisor).

Figures 3 - A provides a local picture of $S_C(\mathbf{R})$ after such a blow up. Locally, this surface is a cylinder. Figure 3 - B is the same as figure 3 - A, but in the universal cover of the cylinder. It shows the geometry of the stable and unstable manifolds of f near p and q . These hyperbolic points are one-sided. Moreover, the multiplier $\lambda(m)$ of f along the unstable manifolds of these points are equal to $\lambda(f)^2$, whereas multipliers of non singular periodic points are equal to $\lambda(f)$. This illustrates a known property of one sided points (see proposition 4.10 in [7]). Last, but not least, the exceptional divisors are heteroclinic connections: Each exceptional divisor is at the same time the stable manifold of one of its periodic points and the unstable manifold of the other one.

Remark 5.11. The existence of such a heteroclinic connection is specific to this construction: If S is an element of Fam and f is a hyperbolic element of \mathcal{A} , there is no heteroclinic connection between periodic points of f in $S(\mathbf{C})$, because if $W^u(p) \setminus \{p\}$ coincides with $W^s(q) \setminus \{q\}$, then $W^u(p) \cup W^s(q)$ would be a one-dimensional compact complex subvariety of \mathbf{C}^3 .

5.3.3. Deformation and periodic points. For any point (A, B, C, D) in \mathcal{H} , all periodic points of $f : S(\mathbf{C}) \rightarrow S(\mathbf{C})$ are real saddle points (property (1) above). As a consequence, we can follow all the periodic points along any deformation of the parameters (A, B, C, D) in \mathcal{H} : If $\alpha(t)$, $t \in [0, 1]$, is an arc of class C^k in \mathcal{H} , and if p_0 is a periodic saddle point of $f : S_{\alpha(0)} \rightarrow S_{\alpha(0)}$ of period N , there exists an arc $p(t)$ of class C^k such that

- (1) for all t , $p(t)$ is contained in $S_{\alpha(t)}$ and $p(0) = p_0$;
- (2) for all t , $p(t)$ is a periodic saddle point of $f : S_{\alpha(t)} \rightarrow S_{\alpha(t)}$ of period N (here we also use the fact that f preserves orientation; otherwise, the period could change when $p(t)$ goes through a singular point of $S_{\alpha(t)}$).

Remark 5.12. The point $p(t)$ is contained in the set $K_{\alpha(t)}(f)$ of points in $S_{\alpha(t)}$ with a bounded f -orbit. The family of compact sets $K_{\alpha(t)}(f)$ depends semi-continuously on t ([5], lemma 3.1), so that the union $\cup_{t \in [0, 1]} K_{\alpha(t)}(f)$ is contained in a fixed compact set \mathcal{K} . The paths $p(t)$, $t \in [0, 1]$, where p describes the set of periodic points of $f : S_{\alpha(0)} \rightarrow S_{\alpha(0)}$ are all contained in \mathcal{K} .

Let us assume that $S_{\alpha(t)}$ is a deformation of the Cayley cubic $S_{\alpha(0)} = S_C$. We shall say that a periodic point w of $S_{\alpha(1)}$ comes from a singular point p of S_C if the deformation $w(t)$ of $w(1) = w$ along $\alpha(t)$ lands at p when $t = 0$.

Lemma 5.13. *Let $m \in S$ be a periodic point of f . The point m is one-sided if and only if it comes from a singular point of the Cayley cubic.*

Proof. Let us denote by $\alpha(t)$ a continuous path in the parameter space such that $S_{\alpha(0)} = S_C$, $S_{\alpha(1)} = S$ and $\alpha(t) \in \mathcal{H}$, for all t in $[0, 1]$; such a path exists by remark 5.3 and theorem 5.2.

Let q be one of the four u -one-sided singular points of S_C . Let us follow q along the deformation and assume that $q(1)$ is not u -one-sided. We can then find a periodic point w in S such that $W^s(w)$ intersects $W^u(q(1))$ in two points a and b , one in each component of $W^u(q(1)) \setminus \{q(1)\}$. We can now follow w and the intersection points a and b along the deformation $S_{\alpha(t)}$. If a tangency between $W^s(w(t))$ and $W^u(q(t))$ occurs at $a(t)$, the order of contact is 2 (see section 5.3.1, property (5)). After the tangency, there are two points of intersection between $W^s(w(t))$ and $W^u(q(t))$, and both of them are contained in $S_{\alpha(t)}(\mathbf{R})$ (see section 5.3.1, property (1)). Between these two points, we choose the one which is closest to $q(t)$ along the component of $W^u(q(t)) \setminus \{q(t)\}$ that contains it, and then continue to follow that point along the deformation. We apply the same strategy to $b(t)$.

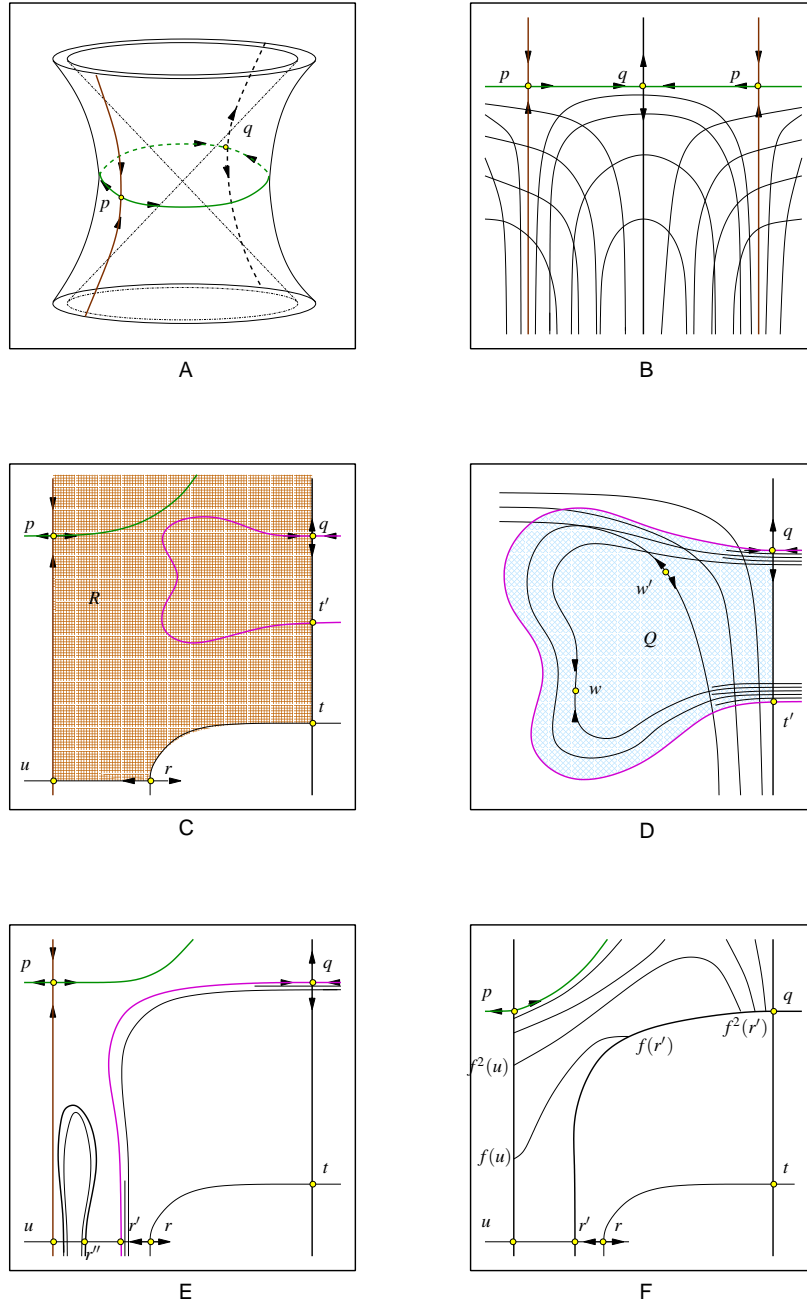


FIGURE 3. Deformation of singularities.

By semi-continuity of the Julia set, $a(t)$ and $b(t)$ stay bounded when t describes $[0, 1]$ (see remark 5.12). Since q is u -one-sided, $a(0)$ and $b(0)$ are

both on the same component of $W^u(q) \setminus \{q\}$. From this we deduce the existence of a parameter t_0 such that $a(t_0) = q(t_0)$ (or $b(t_0) = q(t_0)$), which implies that there is a saddle connection between $w(t_0)$ and $q(t_0)$. This contradicts remark 5.11.

This argument shows that u -one-sided (resp. s -one-sided) singular points of S_C remain u -one-sided (resp. s -one-sided) along any smooth deformation $\alpha(t)$ in \mathcal{H} . The same argument shows that the non-singular periodic points of S_C cannot become one-sided. \square

5.3.4. Deformation and stable manifolds. Next steps aim at giving a description of $K(f)$ and are not absolutely necessary to prove the uniform hyperbolicity. We shall make use of figures 3-B, to E; they represent the geometry of stable and unstable manifolds near p and q after deformation of the Cayley cubic.

Let us study the topology of stable and unstable manifolds of f on a connected deformation $S(\mathbf{R})$ of $S_C(\mathbf{R})$. For this, we consider one exceptional divisor E of $\widehat{S_C}$ and the two fixed points p and q of f on E . Permuting p and q if necessary, we know that q is u -one sided, half of its real unstable manifold going to infinity, and p is s -one sided (see picture 3-B). We fix a periodic point r in S_C which is close to the stable manifold of p : The local unstable manifold of r intersects transversally the stable manifold of p at u and its stable manifold intersects transversally the unstable manifold of q at t , as in figure 3-C. Changing f in one of its iterates, we assume that r is a fixed point.

We follow this picture along a small deformation $S_{\alpha(t)}$ between S_C and $S = S_{\alpha(1)}$, keeping the same local geometry for $W^s(p)$, $W^u(r)$, $W^s(r)$, and $W^u(q)$.

Let $R \subset S(\mathbf{R})$ be the closed region which is bounded by the half of $W^s(p) \setminus \{u\}$ that contains u , the segment of $W^u(r)$ between u and r , the segment of $W^s(r)$ that joins r to t , and the half of $W^u(q) \setminus \{t\}$ that contains t , (see figure 3-C). Let $W_+^s(q)$ be the connected component of $W^s(q) \setminus \{q\}$ which enters R : this half stable manifold is parametrized by $\xi : \mathbf{R}^+ \rightarrow S(\mathbf{R})$, with $\xi(0) = q$ and $\xi(z) \in R$ for small positive real numbers z . The closure of the stable manifold of q covers the set $K(f)$. As a consequence, we may assume that $W_+^s(q) \setminus \{q\}$ exits the region R (in particular, there exists a positive z such that $\xi(z)$ is on ∂R).

Lemma 5.14. *The half stable manifold $W_+^s(q)$ exits R through $W^u(r)$, in between r and u .*

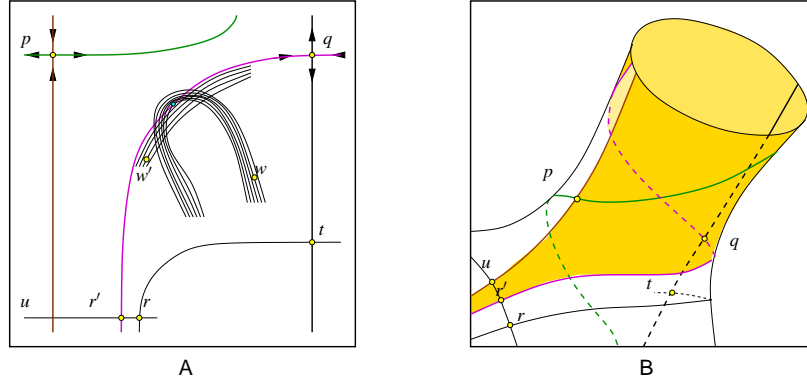
Proof. The stable manifold $W^s(q)$ must exit R through a piece of unstable manifold. We may therefore suppose that it leaves R through $W^u(q)$, in between q and the point t (see picture 3 - C and D). Let t' be the first point of intersection of $W^s(q)$ and the segment $[q, t] \subset W^u(q)$, and let Q be the subset of R which is bounded by the arcs of $W_+^s(q)$ and $W^u(q)$ in between q and t .

Since q is u -one-sided, and since the stable manifolds form a lamination near t' (see section 5.3.1, property (4)), we know that there is no stable manifold that approaches t' along the segment $[t', t] \subset W^u(q)$. On the other hand, t' is not isolated in $K(f)$, and, in particular, there are stable manifolds that cross $W^u(q)$ in between q and t' , and arbitrarily near t' . Those manifolds "follow" $W_+^s(q)$: They enter Q near t' and then exit Q near q (see picture 3 - D). Let now W be an unstable manifold which intersects $W_+^s(q)$ in between q and t' : Such a manifold enters Q , and then has to leave it, intersecting $W_+^s(q)$ a second time between q and t' . These two observations and the density of periodic points in $K(f)$ imply that Q contains periodic points w and w' of f such that the connected component of $W^u(w) \cap Q$ containing w intersects the connected component of $W^s(w') \cap Q$ containing w' in at least two distinct points.

We now follow this picture along a deformation $\alpha(\varepsilon)$ between $S = S_{\alpha(1)}$ and the Cayley cubic $S_C = S_{\alpha(0)}$, as in lemma 5.13. Periodic points, stable and unstable manifolds, and intersections between these curves move continuously along the deformation. In particular, the periodic points $w(\varepsilon)$ and $w'(\varepsilon)$, and the (at least) two intersection points of their stable/unstable manifolds stay in the region $R(\varepsilon)$. Let N be a common period for w and w' . The number of points in the region $R(0)$ on the Cayley cubic with period N is finite, all of them are periodic saddle points (except for p and q) and the local stable and unstable manifolds of these points in the region $R(0)$ intersect in exactly one point (see the proof of lemma 5.13 and picture 3 - B). From this we deduce the existence of a parameter $\varepsilon \geq 0$ for which $W^u(w(\varepsilon)) \cap W^s(w'(\varepsilon))$ is not contained in $S_{\alpha(\varepsilon)}(\mathbf{R})$: At least one intersection points has become complex. Once again, this contradicts the fact that $f_{\mathbf{R}} : S_{\alpha(\varepsilon)}(\mathbf{R}) \rightarrow S_{\alpha(\varepsilon)}(\mathbf{R})$ has maximal entropy (see section 5.3.1, property (1)).

This shows that $W^s(q)$ cannot leave the region R through $W^u(q)$. The only remaining possibility is that $W^s(q)$ leaves R through $W^u(r)$, in between r and u . \square

5.3.5. Deformation, stable manifolds and doubly one-sided points. Let I be the closed segment $[r, u] \subset W^u(r)$. Let r' be the first point of intersection of $W_+^s(q)$ with I (see figure 3-E). Since q is u -one-sided, we know that there is

FIGURE 4. Complement of $K^+(f)$.

no stable manifolds approaching r' from the left. We can therefore define r'' to be the unique point in I which is between u and r' , is contained in $K(f)$, and is closest to r' with these properties.

If r'' is different from u , the stable manifold through r'' enters R and cannot intersect $W^s(q)$ and $W^s(p)$. It must therefore exit R through the interval I , in between r'' and u (see picture 3 - E). We then obtain a contradiction along the same line as in lemma 5.14. This implies that r'' coincides with the point u , and no stable manifold crosses I between r' and u .

The segments $f^n[r', u]$ join the endpoints $f^n(u)$, which converge to p along $W^s(p)$, to $f^n(r')$, which converge to q along $W^s(q)$. These segments are pieces of unstable manifolds and, as such, they can not intersect $W^u(p)$. This shows that the connected component of $W^u(p)$ that enters R does not leave R : This half unstable manifold has to go to infinity, and p is both u and s -one-sided (see picture 3 - F).

5.3.6. Deformation and the geometry of $K(f)$. We can apply the same argument to understand the geometry of stable and unstable manifolds near p . Part B of figure 5.3.5 summarizes our knowledge of the geometry of stable and unstable manifolds near the points p and q after a small deformation of the Cayley cubic: p and q are both u and s -one-sided, and the colored region is contained in the complement of $K(f)$.

Let us now consider a large deformation $S_{\alpha(t)}$ of the Cayley cubic S_C . Following p , u , r , t , q , and the stable/unstable manifolds of these points along the deformation, we can follow the region R along $\alpha(t)$. Since there is no saddle connection in $S_{\alpha(t)}$ for $t \neq 0$, the geometry of R with respect to local stable and unstable manifolds in R does not change. The results

obtained above for small deformation remain therefore valid for arbitrarily large deformation $\alpha(t) \subset \mathcal{H}$.

5.3.7. Absence of tangency and hyperbolicity. Let us assume that there is at least one set of parameters (A, B, C, D) , for which $S(\mathbf{R})$ is connected and $f_{\mathbf{R}}$ is not uniformly hyperbolic along $K(f)$. Then, there is a tangency between the stable manifold of a u -one-sided periodic point q and an unstable manifold. Iterating f , we can find such tangencies in arbitrarily small neighborhoods of q . Since q is u -one-sided, the previous steps describe the geometry of the stable and unstable manifolds near q . Figure 5.3.5-A represents such a possible tangency, and it shows the geometry of the unstable lamination near such a tangency (see [8], picture 4.1 and sections 3 and 4, for a detailed description).

In an arbitrarily small neighborhood \mathcal{U} of the tangency, we can find a periodic saddle point w , such that the connected component $W_{loc}^u(w)$ of $W^u(w) \cap \mathcal{U}$ containing w intersects $W^s(q)$ in two points. Then, we can find a second periodic saddle point w' such that $W^s(w')$ intersects $W_{loc}^u(w)$ in two points (see figure 5.3.5-A).

Since $S(\mathbf{R})$ is connected, we can follow this picture along a deformation $\alpha(\varepsilon) \in \mathcal{H}$, $\varepsilon \in (0, 1]$, which approaches the Cayley parameters $(0, 0, 0, 4)$ when ε goes to 0 (see remark 5.3). We know from section 5.3.3 that the periodic saddle points $r(\varepsilon)$, $t(\varepsilon)$, $u(\varepsilon)$, $w(\varepsilon)$ and $w'(\varepsilon)$, move continuously along the deformation. From sections 5.3.5 and 5.3.6, we can also assume that the geometry of the stable and unstable manifolds of $r(\varepsilon)$, $t(\varepsilon)$, $q(\varepsilon)$ and $p(\varepsilon)$ remains the same along the deformation; in particular, since the periodic points $w(\varepsilon)$ and $w'(\varepsilon)$ cannot cross the stable or unstable manifolds of other periodic points during the deformation, they both stay in the interior of the region $R(\varepsilon)$. We then get a contradiction as in section 5.3.5.

Since there is no tangency, the dynamics of f is uniformly hyperbolic on $K(f)$. This proves theorem 5.10.

5.4. Strips and bounded orbits. Let (A, B, C, D) be an element of \mathcal{H} . Let f be a hyperbolic element of \mathcal{A} . The surface $S(\mathbf{R})$ defined by this set of parameters is connected, and $f : S(\mathbf{R}) \rightarrow S(\mathbf{R})$ is uniformly hyperbolic on $K(f)$, so that we can apply proposition 2.1.1 of [11]: The set

$$W_{\mathbf{R}}^s(K(f)) = K^+(f) \cap S(\mathbf{R})$$

is laminated by stable manifolds of points in $K(f)$; if a point m in $K^+(f)$ is on the boundary of the complement of $W_{\mathbf{R}}^s(K(f))$, then m is on the stable manifold of a periodic u -one-sided periodic point of f . From section 5.3.3,

we know that f has exactly eight periodic one-sided points, each of them coming from a singularity of the Cayley cubic. From sections 5.3.4 and 5.3.6, the stable manifolds of the two one-sided points coming from one singularity bound a strip, as in picture 5.3.5-B. This proves the following result, which was first numerically observed by Catarino and MacKay (see [18], page 61 for example), and "explains" pictures 2-A,C.

Theorem 5.15 (MacKay observation). *If $S(\mathbf{R})$ is connected, f has exactly eight one-sided fixed points $p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4$. All of them come from singularities of the Cayley cubic by deformation; all of them are both u and s -one-sided. Moreover, the stable manifolds of p_i and q_i ($i = 1, 2, 3, 4$) bound an open strip homeomorphic to $\mathbf{R} \times (-1, 1)$, and the complement of $K^+(f) \cap S(\mathbf{R})$ coincides with the union of these four strips.*

Remark 5.16. We shall prove in theorem 6.5 that the Hausdorff dimension of $K^+(f) \cap S(\mathbf{R})$ is strictly less than 2. In particular, its complement, i.e. the union of the four strips, has full Lebesgue measure ; almost all orbits go to infinity under iteration of f . The same is true for the complement of $K^+(f)$ in $S(\mathbf{C})$.

6. SCHRÖDINGER OPERATORS AND PAINLEVÉ EQUATIONS

6.1. Discrete Schrödinger operators. Let us now apply the previous results to the study of the spectrum of certain discrete Schrödinger operators. There is a huge literature on the subject, and we refer to [19] and [20] for background results and a short bibliography.

6.1.1. Discrete Schrödinger operators and substitutions. Let W^* be the set of finite words in the letters a and b . Let $\mathfrak{t} : \{a, b\} \rightarrow W^* \setminus \{\emptyset\}$ be a substitution. In what follows, we will assume that \mathfrak{t} is invertible, which means that \mathfrak{t} extends to an automorphism $\Phi_{\mathfrak{t}}$ of the free group $F_2 = \langle a, b \mid \emptyset \rangle$, and that \mathfrak{t} is primitive, which means that $\Phi_{\mathfrak{t}}$ is hyperbolic ; in other words, the image of $\Phi_{\mathfrak{t}}$ in $\text{Out}(F_2) = \text{GL}(2, \mathbf{Z})$ is a hyperbolic matrix, with two distinct eigenvalues $\lambda_+(\mathfrak{t})$ and $\lambda_-(\mathfrak{t})$ satisfying

$$|\lambda_+(\mathfrak{t})| = |1/\lambda_-(\mathfrak{t})| > 1.$$

Under these hypotheses, there is a unique infinite word u_+ in the two letters a and b such that $\mathfrak{t}(u_+) = u_+$.

Example 6.1. The Fibonacci substitution \mathfrak{t}_F , defined by $\mathfrak{t}_F(a) = b$ and $\mathfrak{t}_F(b) = ba$, provides a good and famous example of such an invertible primitive substitution. Its fixed word starts with *babbababbabbabbabbabbabba...*

Let W be the set of bi-infinite words in a and b and $\tilde{T} : W \rightarrow W$ be the left shift. Let \tilde{u}_+ be any completion of u_+ on the left. We then define Ω to be the ω -limit set of the \tilde{T} -orbit of \tilde{u}_+ :

$$\Omega = \{v \in W \mid \text{there exists a sequence } n_i \rightarrow +\infty, \text{ such that } \tilde{T}^{n_i}(\tilde{u}_+) \rightarrow v\}.$$

Since \mathfrak{t} is primitive, the restriction of the left shift \tilde{T} to the set Ω is a minimal and uniquely ergodic homeomorphism $T : \Omega \rightarrow \Omega$. The unique T -invariant probability measure on Ω will be denoted by ν .

Remark 6.2. The subshift $T : \Omega \rightarrow \Omega$ encodes the dynamics of a rotation $R_\alpha : \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{R}/\mathbf{Z}$, where α is a quadratic integer (see [1]). This provides a measurable conjugation between R_α and T which sends the Lebesgue measure dx to ν .

Let us now fix an element w in Ω , and define the potential $V_w : \mathbf{Z} \rightarrow \mathbf{R}$ by $V_w(n) = 1$ if $w_n = a$ and $V_w(n) = 0$ if $w_n = b$. Let κ be any complex number (κ is the so called "coupling parameter"). If $(\xi(n))_{n \in \mathbf{Z}}$ is a complex valued sequence, we define

$$H_{\kappa,w}(\xi)(n) = \xi(n+1) + \xi(n-1) + \kappa V_w(n) \xi(n).$$

The discrete Schrödinger operator $H_{\kappa,w}$ induces a bounded linear operator on $l^2(\mathbf{Z})$, with norm at most $2 + |\kappa|$. The adjoint of $H_{\kappa,w}$ is $H_{\bar{\kappa},w}$, so that $H_{\kappa,w}$ is self-adjoint if and only if κ is a real number.

6.1.2. Almost sure spectrum and Lyapunov exponent. Since T is ergodic with respect to ν , there exists a subset Σ_κ of \mathbf{C} (of \mathbf{R} if κ is real) such that the spectrum of $H_{\kappa,w} : l^2(\mathbf{Z}) \rightarrow l^2(\mathbf{Z})$ coincides with Σ_κ for ν -almost all w in Ω . This set is the "almost sure spectrum" of the family $H_{\kappa,w}$.

To understand the spectrum of $H_{\kappa,w}$, one is led to solve the eigenvalue equation $H_{\kappa,w}(\xi) = E\xi$ (E in \mathbf{R} or \mathbf{C}). For any initial condition $(\xi(0), \xi(1))$, there is a unique solution, which is given by the recursion formula

$$\begin{pmatrix} \xi(n+1) \\ \xi(n) \end{pmatrix} = \begin{pmatrix} E - \kappa V_w(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi(n) \\ \xi(n-1) \end{pmatrix}, \quad n \in \mathbf{Z}.$$

Let $M_{\kappa,E} : W^* \rightarrow \text{SL}(2, \mathbf{C})$ be defined by

$$M_{\kappa,E}(a) = \begin{pmatrix} E - \kappa & -1 \\ 1 & 0 \end{pmatrix}, \quad M_{\kappa,E}(b) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix},$$

and by

$$M_{\kappa,E}(u_1 \dots u_n) = \prod_{i=0}^{n-1} M_{\kappa,E}(u_{n-i})$$

for any word $u = u_1 \dots u_n$ of length n . This defines a $\text{SL}(2, \mathbf{C})$ -valued cocycle over the dynamical system (Ω, T, ν) . Applying Osseledets' theorem, each

choice of a coupling parameter κ and an energy E gives rise to a non negative Lyapunov exponent $\gamma(\kappa, E)$, such that

$$\begin{aligned}\gamma(\kappa, E) &= \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\Omega} \log \|M_{\kappa, E}(w_1 w_2 \dots w_{n-1})\| d\nu(w) \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M_{\kappa, E}(w_1 w_2 \dots w_{n-1})\|,\end{aligned}$$

for ν -almost all w in Ω . The Lyapunov function $\gamma(\kappa, E)$ is linked to the almost sure spectrum Σ_{κ} by the following result.

Theorem 6.3 (see [19]). *Let κ be a real number. The almost sure spectrum Σ_{κ} coincides with the set of energies for which the Lyapunov exponent vanishes.*

6.1.3. Trace map dynamics, Lyapunov exponent, and Hausdorff dimension. Let us fix the coupling parameter κ . Let $S_{4+\kappa^2}$ be the character surface $x^2 + y^2 + z^2 - xyz = 4 + \kappa^2$. The Schrödinger curve of $S_{4+\kappa^2}$ is the parametrized rational curve $s : \mathbf{C} \rightarrow S_{4+\kappa^2}$, which is defined by $s(E) = (x(E), y(E), z(E))$, with

$$\begin{aligned}(x(E), y(E), z(E)) &= (\text{tr}(M_{\kappa, E}(a)), \text{tr}(M_{\kappa, E}(b)), \text{tr}(M_{\kappa, E}(ab))) \\ &= (E - \kappa, E, E(E - \kappa) - 2).\end{aligned}$$

Remark 6.4. The intersection of $S_{4+\kappa^2}$ with the plane $y = x + \kappa$ is a reducible cubic curve: It is the union of $s(\mathbf{C})$ with the line $\{z = 2, y = x + \kappa\}$; the involution s_z permutes these two curves.

Let $f_{\mathbf{t}}$ be the polynomial automorphism of $S_{4+\kappa^2}$ which is determined by the automorphism $(\Phi_{\mathbf{t}})^{-1} : F_2 \rightarrow F_2$. By definition of $f_{\mathbf{t}}$, we have

$$(\text{tr}(M_{\kappa, E}(\mathbf{t}(a))), \text{tr}(M_{\kappa, E}(\mathbf{t}(b))), \text{tr}(M_{\kappa, E}(\mathbf{t}(ab)))) = f_{\mathbf{t}}(s(E)).$$

In [19], Damanik proved that $\gamma(\kappa, E)$ vanishes if and only if the point $s(E)$ has a bounded forward $f_{\mathbf{t}}$ -orbit. In other words, Σ_{κ} is given by the intersection between the Schrödinger curve and the set $K^+(f_{\mathbf{t}})$:

$$\Sigma_{\kappa} = \{E \in \mathbf{C} \mid s(E) \in K^+(f_{\mathbf{t}})\}. \quad (6.1)$$

We can now apply MacKay observation, i.e. theorem 5.15, which tells us that the complement of $s(\Sigma_{\kappa})$ in the real Schrödinger curve is obtained by intersecting $s(\mathbf{R})$ with the four strips associated to the one-sided points of f . This means that *gaps in the complement of the spectrum are bounded by intersection points between $s(\mathbf{R})$ and the eight curves $W^s(q_i)$ and $W^s(p_i)$, $i = 1, 2, 3$, and 4).*

Theorem 6.5. *The Hausdorff dimension of Σ_κ , $\kappa \in \mathbf{R}$, is a real analytic function of κ^2 . Moreover,*

$$0 < \text{Haus}(\Sigma_\kappa) \leq 1, \quad \forall \kappa \in \mathbf{R},$$

and $\text{Haus}(\Sigma_\kappa) = 1$ if and only if $\kappa = 0$.

This statement confirms numerical observations that can be found, for example, in [38] and [37]; it is stronger than the fact that Σ_κ has zero Lebesgue measure when $\kappa \neq 0$, a property which was proved by Kotani in the eighties (see [20]). Here, it appears as a corollary of results in dynamical systems which are due to Bowen, Pesin, and Ruelle.

Proof. When κ is a non zero real number, we obviously have $4 + \kappa^2 > 4$, and theorem 1.2 shows that the dynamics of f_1 is uniformly hyperbolic on its Julia set. By results of Hasselblatt [32], the stable and unstable distributions of f_1 are smooth, and the holonomy maps between two transversals of the stable (resp. unstable) laminations are Lipschitz continuous. In particular, the Hausdorff dimension of the sets

$$W_{loc}^u(m) \cap K^+(f_1)$$

does not depend on the choice of m in $K(f_1)$, and, by (6.1), it coincides with the dimension of Σ_κ . The map f_1 is area-preserving: As in [48], corollary 4.7, this implies that the Hausdorff dimension of the sets $W_{loc}^s(m) \cap K^-(f_1)$ coincides also with $\text{Haus}(\Sigma_\kappa)$.

Using Bowen-Ruelle thermodynamic formalism, as it is done in [47], we obtain that the Hausdorff dimension of Σ_κ is an analytic function of κ^2 . Since the function $G_{f_1|s(E)}^+$ is Hölder continuous the Hausdorff dimension of Σ_κ is strictly positive.

Let us now show that $\text{Haus}(\Sigma_\kappa)$ is strictly less than 1. If $\text{Haus}(\Sigma_\kappa) = 1$, the Hausdorff dimension of the slices $W_{loc}^u(m) \cap K^+(f_1)$ and $W_{loc}^s(m) \cap K^-(f_1)$ are also equal to 1. Theorem 22.1 of [43] then shows that the Lebesgue measure of these sets is strictly positive. By Hasselblatt's result, the Lebesgue measure of $K(f_1)$ is positive, and by Bowen-Ruelle's theorem ([14], theorem 5.6), the set $K(f_1)$ must be an attractor of $f_1 : S_{4+\kappa^2} \rightarrow S_{4+\kappa^2}$. This contradicts the fact that $K(f_1)$ is compact and f is area preserving. \square

Remark 6.6. It would be interesting to settle a complete dictionary between dynamics of the trace map and properties of the spectrum. For example, the Green function of f_1 should coincide with the Lyapunov function $\gamma(\kappa, E)$ along the Schrödinger curve ; together with Thouless formula, this would identify the density of states dk_κ with the measure obtained by slicing $T_{f_1}^+$

with the Schrödinger curve: $dk_K = s^*(T_{f_1}^+)$ (see [45] for related results, the definition of dk_K , and Thouless formula).

6.2. Monodromy of Painlevé VI equation. The sixth Painlevé equation $P_{VI} = P_{VI}(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$ is the second order non linear ordinary differential equation

$$P_{VI} \quad \left\{ \begin{aligned} \frac{d^2 q}{dt^2} &= \frac{1}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) \left(\frac{dq}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right) \left(\frac{dq}{dt} \right) \\ &+ \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left(\frac{\theta_\delta^2}{2} - \frac{\theta_\alpha^2}{2} \frac{t}{q^2} + \frac{\theta_\beta^2}{2} \frac{t-1}{(q-1)^2} + \frac{1-\theta_\gamma^2}{2} \frac{t(t-1)}{(q-t)^2} \right). \end{aligned} \right.$$

the coefficients of which depend on complex parameters $\theta = (\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$.

As explained in [36] (see also [16]), the monodromy of Painlevé equation provides a representation of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, t_0)$ into the group of analytic diffeomorphisms of the space of initial conditions $(q(t_0), q'(t_0))$ (see [36] for a precise necessary description of this space). Via the Riemann-Hilbert correspondence, the space of initial conditions is analytically isomorphic to (a desingularization of) $S_{(A,B,C,D)}$ with parameters

$$a = 2 \cos(\pi \theta_\alpha), \quad b = 2 \cos(\pi \theta_\beta), \quad c = 2 \cos(\pi \theta_\gamma), \quad d = 2 \cos(\pi \theta_\delta), \quad (6.2)$$

the monodromy action on the space of initial conditions is conjugate to the action of Γ_2 on the surface $S_{(A,B,C,D)}$.

From this, and from sections 5.3 and 6, we deduce the following result, thereby answering a recent question raised by Iwasaki and Uehara, as problem 15 of [35].

Theorem 6.7. *Let $(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$ be parameters of Painlevé sixth equation such that*

- (i) *for $\varepsilon = \alpha, \beta, \gamma$, and δ , the real part of Θ_ε is an integer n_ε , and*
- (ii) *$n_\alpha + n_\beta + n_\gamma + n_\delta$ is odd.*

Let η be any loop in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and let $f_\eta : S_{(A,B,C,D)} \rightarrow S_{(A,B,C,D)}$ be the monodromy transformation defined by η (through Riemann-Hilbert correspondence). If the entropy of f_η is positive, then

- *all periodic points of f_η are contained in the real part $S_{(A,B,C,D)}(\mathbf{R})$ of the surface;*
- *the Hausdorff-dimension of the set of bounded f_η -orbits is less than 2;*
- *the unique invariant probability measure of maximal entropy μ_{f_η} is singular with respect to the Lebesgue measure on $S_{(A,B,C,D)}(\mathbf{R})$ (and $S_{(A,B,C,D)}(\mathbf{C})$).*

This theorem should be compared to Goldman's results regarding ergodic properties of the whole Γ_2^* action with respect to the invariant area form Ω (see the definition of Ω in section 2.2). As a particular case of Goldman's theorem, the action of Γ_2^* on $S_D(\mathbf{R})$ is ergodic with respect to Ω if, and only if $4 < D \leq 20$ (see [30]). Another interesting example is given by the Markoff surface S_0 . In this example, the quasifuchsian space \mathbf{QF} provides an open invariant subset of $S_0(\mathbf{C})$. This shows that the action of Γ_2 on $S_0(\mathbf{C})$ is not ergodic. Theorem 6.7 and these results suggest that, for most parameters, the dynamics of the monodromy of Painlevé equation is not correctly described by the invariant area form Ω .

REFERENCES

- [1] Pierre Arnoux. Sturmian sequences. In *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Math.*, pages 143–198. Springer, Berlin, 2002.
- [2] Eric Bedford and Jeffrey Diller. Energy and invariant measures for birational surface maps. *Duke Math. J.*, 128(2):331–368, 2005.
- [3] Eric Bedford, Mikhail Lyubich, and John Smillie. Distribution of periodic points of polynomial diffeomorphisms of \mathbf{C}^2 . *Invent. Math.*, 114(2):277–288, 1993.
- [4] Eric Bedford, Mikhail Lyubich, and John Smillie. Polynomial diffeomorphisms of \mathbf{C}^2 . IV. The measure of maximal entropy and laminar currents. *Invent. Math.*, 112(1):77–125, 1993.
- [5] Eric Bedford and John Smillie. Polynomial diffeomorphisms of \mathbf{C}^2 : currents, equilibrium measure and hyperbolicity. *Invent. Math.*, 103(1):69–99, 1991.
- [6] Eric Bedford and John Smillie. Polynomial diffeomorphisms of \mathbf{C}^2 . II. Stable manifolds and recurrence. *J. Amer. Math. Soc.*, 4(4):657–679, 1991.
- [7] Eric Bedford and John Smillie. Polynomial diffeomorphisms of \mathbf{C}^2 . VIII. Quasi-expansion. *Amer. J. Math.*, 124(2):221–271, 2002.
- [8] Eric Bedford and John Smillie. Real polynomial diffeomorphisms with maximal entropy: Tangencies. *Ann. of Math. (2)*, 160(1):1–26, 2004.
- [9] Robert L. Benedetto and William M. Goldman. The topology of the relative character varieties of a quadruply-punctured sphere. *Experiment. Math.*, 8(1):85–103, 1999.
- [10] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J., 1974. Annals of Mathematics Studies, No. 82.
- [11] Christian Bonatti and Rémi Langevin. Difféomorphismes de Smale des surfaces. *Astérisque*, (250):viii+235, 1998. With the collaboration of E. Jeandenans.
- [12] Brian H. Bowditch. Markoff triples and quasi-Fuchsian groups. *Proc. London Math. Soc. (3)*, 77(3):697–736, 1998.
- [13] Rufus Bowen. Entropy and the fundamental group. In *The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977)*, volume 668 of *Lecture Notes in Math.*, pages 21–29. Springer, Berlin, 1978.
- [14] Rufus Bowen and David Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.*, 29(3):181–202, 1975.
- [15] Serge Cantat. Dynamique des automorphismes des surfaces $K3$. *Acta Math.*, 187(1):1–57, 2001.

- [16] Serge Cantat and Frank Loray. Character varieties, dynamics and Painlevé VI. *preprint*, pages 1–63, 2007.
- [17] Martin Casdagli. Symbolic dynamics for the renormalization map of a quasiperiodic Schrödinger equation. *Comm. Math. Phys.*, 107(2):295–318, 1986.
- [18] Nuno Ricardo Catarino. Quantum statistical mechanics of generalised Frenkel-Kantorova models. *PhD, Warwick University*, pages 1–69, 2004.
- [19] David Damanik. Substitution Hamiltonians with bounded trace map orbits. *J. Math. Anal. Appl.*, 249(2):393–411, 2000.
- [20] David Damanik. Lyapunov exponents and spectral analysis of ergodic Schrödinger operators: A survey of Kotani theory and its applications. *preprint*, 2006.
- [21] Jeffrey Diller and Charles Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.*, 123(6):1135–1169, 2001.
- [22] Tien-Cuong Dinh and Nessim Sibony. Green currents for holomorphic automorphisms of compact Kähler manifolds. *J. Amer. Math. Soc.*, 18(2):291–312 (electronic), 2005.
- [23] Boris Dubrovin and Marta Mazzocco. Monodromy of certain Painlevé-VI transcendents and reflection groups. *Invent. Math.*, 141(1):55–147, 2000.
- [24] Romain Dujardin. Laminar currents and birational dynamics. *Duke Math. J.*, 131(2):219–247, 2006.
- [25] Marat H. Èl'-Huti. Cubic surfaces of Markov type. *Mat. Sb. (N.S.)*, 93(135):331–346, 487, 1974.
- [26] Charles Favre. Classification of 2-dimensional contracting rigid germs and Kato surfaces. I. *J. Math. Pures Appl. (9)*, 79(5):475–514, 2000.
- [27] John Erik Fornæss and Nessim Sibony. Complex Hénon mappings in \mathbb{C}^2 and Fatou-Bieberbach domains. *Duke Math. J.*, 65(2):345–380, 1992.
- [28] William M. Goldman. Topological components of spaces of representations. *Invent. Math.*, 93(3):557–607, 1988.
- [29] William M. Goldman. Ergodic theory on moduli spaces. *Ann. of Math. (2)*, 146(3):475–507, 1997.
- [30] William M. Goldman. The modular group action on real $SL(2)$ -characters of a one-holed torus. *Geom. Topol.*, 7:443–486 (electronic), 2003.
- [31] William M. Goldman. Mapping class group dynamics on surface group representations. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 189–214. Amer. Math. Soc., Providence, RI, 2006.
- [32] Boris Hasselblatt. Regularity of the Anosov splitting and of horospheric foliations. *Ergodic Theory Dynam. Systems*, 14(4):645–666, 1994.
- [33] John Hubbard, Peter Papadopol, and Vladimir Veselov. A compactification of Hénon mappings in \mathbb{C}^2 as dynamical systems. *Acta Math.*, 184(2):203–270, 2000.
- [34] John H. Hubbard and Ralph W. Oberste-Vorth. Hénon mappings in the complex domain. I. The global topology of dynamical space. *Inst. Hautes Études Sci. Publ. Math.*, (79):5–46, 1994.
- [35] Katsunori Iwasaki and Takato Uehara. Chaos in the sixth Painlevé equation. In *Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies*, RIMS Kôkyûroku Bessatsu, B2, pages 73–88. Res. Inst. Math. Sci. (RIMS), Kyoto, 2007.
- [36] Katsunori Iwasaki and Takato Uehara. An ergodic study of Painlevé VI. *Math. Ann.*, 338(2):295–345, 2007.

- [37] Mahito Kohmoto. Metal-insulator transition and scaling for incommensurate systems. *Phys. Rev. Lett.*, 51(13):1198–1201, 1983.
- [38] Mahito Kohmoto, Leo P. Kadanoff, and Chao Tang. Localization problem in one dimension: mapping and escape. *Phys. Rev. Lett.*, 50(23):1870–1872, 1983.
- [39] Curtis T. McMullen. *Renormalization and 3-manifolds which fiber over the circle*, volume 142 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.
- [40] Yair N. Minsky. The classification of punctured-torus groups. *Ann. of Math. (2)*, 149(2):559–626, 1999.
- [41] Yair N. Minsky. End invariants and the classification of hyperbolic 3-manifolds. In *Current developments in mathematics, 2002*, pages 181–217. Int. Press, Somerville, MA, 2003.
- [42] Sheldon E. Newhouse. Continuity properties of entropy. *Ann. of Math. (2)*, 129(2):215–235, 1989.
- [43] Yakov B. Pesin. *Dimension theory in dynamical systems*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1997. Contemporary views and applications.
- [44] John A. G. Roberts. Escaping orbits in trace maps. *Phys. A*, 228(1-4):295–325, 1996.
- [45] Christophe Sabot. Integrated density of states of self-similar Sturm-Liouville operators and holomorphic dynamics in higher dimension. *Ann. Inst. H. Poincaré Probab. Statist.*, 37(3):275–311, 2001.
- [46] Nessim Sibony. Dynamique des applications rationnelles de \mathbb{P}^k . In *Dynamique et géométrie complexes (Lyon, 1997)*, volume 8 of *Panor. Synthèses*, pages ix–x, xi–xii, 97–185. Soc. Math. France, Paris, 1999.
- [47] Alberto Verjovsky and He Wu. Hausdorff dimension of Julia sets of complex Hénon mappings. *Ergodic Theory Dynam. Systems*, 16(4):849–861, 1996.
- [48] Christian Wolf. Hausdorff and topological dimension for polynomial automorphisms of \mathbb{C}^2 . *Ergodic Theory Dynam. Systems*, 22(4):1313–1327, 2002.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE RENNES, RENNES, FRANCE
 E-mail address: serge.cantat@univ-rennes1.fr